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X-RAYS AS A BRANCH OF OPTICS

By Arthur H. Compton

One of the most fascinating aspects of recent physics research has been the gradual extension of the familiar laws of optics to the very high frequencies of x-rays, until at present there is hardly a phenomenon in the realm of light whose parallel is not found in the realm of x-rays. Reflection, refraction, diffuse scattering, polarization, diffraction, emission and absorption spectra, photoelectric effect, all of the essential characteristics of light have been found also to be characteristic of x-rays. At the same time it has been found that some of these phenomena undergo a gradual change as we proceed to the extreme frequencies of x-rays, and as a result of these interesting changes in the laws of optics we have gained new information regarding the nature of light.

It has not always been recognized that x-rays is a branch of optics. As a result of the early studies of Röntgen and his followers it was concluded that x-rays could not be reflected or refracted, that they were not polarized on traversing crystals, and that they showed no signs of diffraction on passing through narrow slits. In fact, about the only property which they were found to possess in common with light was that of propagation in straight lines. Many will recall also the heated debate between Barkla and Bragg, as late as 1910, one defending the idea that x-rays are waves like light, the other that they consist of streams of little bullets called "neutrons." It is a debate on which the last word has not yet been said!

THE REFRACTION AND REFLECTION OF X-RAYS

We should consider the phenomena of refraction and reflection as one problem, since it is a well known law of optics that reflection can occur

only from a boundary surface between two media of different indices of refraction. If one is found, the other must be present.

In his original examination of the properties of x-rays, Röntgen[1] tried unsuccessfully to obtain refraction by means of prisms of a variety of materials such as ebonite, aluminium and water. Perhaps the experiment of this type most favorable for detecting refraction was one by Barkla.[2] In this work x-rays of a wave length which excited strongly the characteristic K radiation from bromine were passed through a crystal of potassium bromide. The precision of his experiment was such that he was able to conclude that the refractive index for a wave length of 0.5A probably differed from unity by less than 5 parts in a million.

Although these direct tests for refraction of x-rays were unsuccessful, Stenström observed[3] that for x-rays whose wave lengths are greater than about 3A, reflected from crystals of sugar and gypsum, Bragg's law, $n\lambda = 2D \sin\theta$, does not give accurately the angles of reflection. He interpreted the difference as due to an appreciable refraction of the x-rays as they enter the crystal. Measurements by Duane and Siegbahn and their collaborators[4] showed that discrepancies of the same type occur, though they are very small indeed, when ordinary x-rays are reflected from calcite.

The direction of the deviations in Stenström's experiments indicated that the index of refraction of the crystals employed was less than unity. If this is the case also for other substances, total reflection should occur when x-rays in air strike a polished surface at a sufficiently sharp glancing angle, just as light in a glass prism is totally reflected from a surface between the glass and air if the light strikes the surface at a sufficiently sharp angle. From a measurement of this critical angle for total reflection, it should be possible to determine the index of refraction of the x-rays.

When the experiment was tried,[5] the results were strictly in accord with these predictions. The apparatus was set up as shown in Fig. 1, reflecting a very narrow sheet of x-rays from a polished mirror onto

[1] W. Röntgen, Sitzungber. der Würzburger Phys. Med. Ges. Jahrg. 1895. These papers are reprinted in German in Ann. d. Phys., *64*, p. 1; 1898, and in English translation by A. Stanton in Science, *3*, p. 227; 1896.

[2] C. G. Barkla, Phil. Mag., *31*, p. 257; 1916.

[3] W. Stenström, Dissertation, Lund, 1919.

[4] Duane and Patterson, Phys. Rev., *16*, p. 532; 1920. M. Siegbahn, C. R., *173*, p. 1350; 1921; *174*, p. 745; 1922.

[5] A. H. Compton, Phil. Mag., *45*, p. 1121; 1923.

the crystal of a Bragg spectrometer. It was found that the beam could be reflected from surfaces of polished glass and silver through several minutes of arc. By studying the spectrum of the reflected beam, the critical glancing angle was found to be approximately proportional to

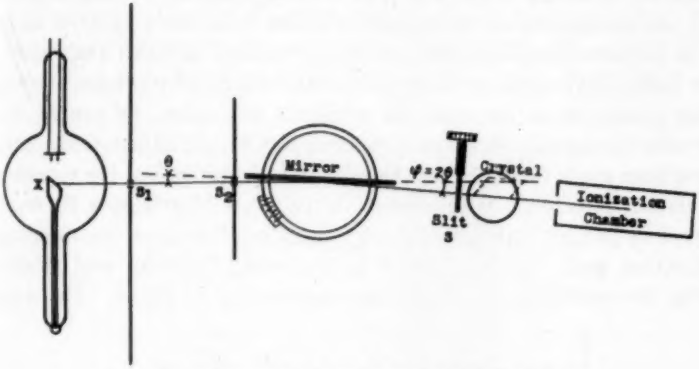

FIG. 1. *Apparatus for studying the total reflection of x-rays.*

the wave length. For ordinary x-rays whose wave length is half an Angström, the critical glancing angle from crown glass was found to be about 4.5 minutes of arc, which means a refractive index differing from unity by a little less than 1 part in a million.

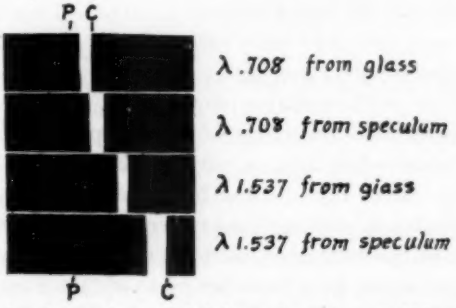

FIG. 2. *Total reflection of x-rays from polished glass and speculum metal (Doan). P = direct beam, C = critical angle of the totally reflected beam.*

Fig. 2 shows some photographs of the totally reflected beam and the critical angle for total reflection taken recently by Dr. Doan[6] working at Chicago. From the sharpness of the critical angles shown in this

[6] R. L. Doan, Phil. Mag., *20*, p. 100; 1927.

figure, it is evident that a precise determination of the refractive index can thus be made.

You will recall that when one measures the index of refraction of a beam of light in a glass prism it is customary to set the prism at the angle for minimum deviation. This is done primarily because it simplifies the calculation of the refractive index from the measured angles. It is an interesting comment on the psychology of habit that most of the earlier investigators of the refraction of x-rays by prisms also used their prisms set at the angle for minimum deviation. Of course, since the effect to be measured was very small indeed, the adjustments should have been made to secure not the minimum deviation but the maximum deviation possible. After almost thirty years of attempts to refract x-rays by prisms, experiments under the conditions to secure maximum refraction were first performed by Larsson, Siegbahn and Waller,[7] using the arrangement shown diagrammatically in Fig. 3. The x-rays

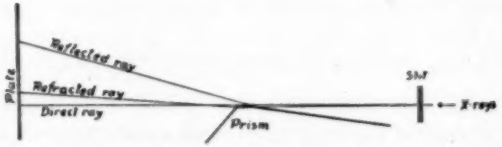

FIG. 3. *Refraction of x-rays by a glass prism. Arrangement of Larsson, Siegbahn and Waller.*

struck the face of the prism at a fine glancing angle, just greater than the critical angle for the rays which are refracted. Thus the direct rays, the refracted rays, and the totally reflected rays of greater wave length were all recorded on the same plate.

Fig. 4 shows one of the resulting photographs. Here we see a complete dispersion spectrum of the refracted x-rays, precisely similar to the spectrum obtained when light is refracted by a prism of glass. The presence of the direct ray and the totally reflected ray on the same plate make possible all the angle measurements necessary for a precise determination of the refractive index for each spectrum line.

For a generation we have been trying to obtain a quantitative test of Drude and Lorentz's dispersion theory in the ordinary optical region. But our ignorance regarding the number and the natural frequency of the electron oscillators in the refractive medium has foiled all such attempts. For the extreme frequencies of x-rays, however, the problem becomes greatly simplified. In the case of substances such as

[7] Larsson, Siegbahn and Waller, Naturwiss, 1924.

glass, the x-ray frequencies are much higher than the natural frequencies of the oscillators in the medium, and the only knowledge which the theory requires is that of the number of electrons per unit volume in the dispersive medium. If we assume the number of electrons per atom to be equal to the atomic number, we are thus able to calculate at once the refractive index of the medium for x-rays. In the case of glass this calculation gives agreement with experiment within the experimental error, which is in some cases less than one per cent. So we may say

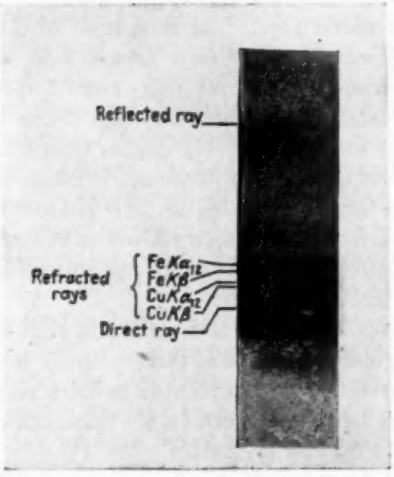

FIG. 4. *Prism spectrum of x-rays obtained by Larsson, Siegbahn and Waller.*

that the laws of optical dispersion given by the electron theory are first established on a quantitative basis by these experiments on the refraction of x-rays.

Another way of looking at the problem is to assume the validity of the dispersion equation developed from the electron theory, and to use these measurements of the refraction of x-rays to calculate the number of electrons in each atom of the refracting material. This affords us what is probably our most direct as well as our most precise means of determining this number. The precision of the experiments is now such that we can say that the number of electrons per atom effective in refracting x-rays is within less than one half of one per cent equal to the atomic number of the atom.

Thus optical refraction and reflection are extended to the region of x-rays, and this extension has brought with it more exact knowledge not only of the laws of optics but also of the structure of the atom.

THE DIFFRACTION OF X-RAYS

Early in the history of x-rays it was recognized that most of the properties of these rays might be explained if, as suggested by Wiechert,[8] they consist of electromagnetic waves much shorter than those of light. Haga and Wind performed a careful series of experiments[9] to detect any possible diffraction by a wedge shaped slit a few thousandths of an inch broad at its widest part. The magnitude of the broadening was about that which would result[10] from rays of 1.3A wave length. The experiments were repeated by yet more refined methods by Walter and Pohl,[11] who came to the conclusion that if any diffraction effects were present, they were considerably smaller than Haga and Wind had estimated. But on the basis of photometric measurements of Walter and Pohl's plates by Koch,[12] using his new photoelectric microphotometer, Sommerfeld found[13] that their photographs indicated an effective wave length for hard x-rays of .4A, and for soft x-rays a wave length measurably greater.

It may have been because of their difficulty that these experiments did not carry as great conviction as their accuracy would seem to have warranted. Nevertheless it was this work perhaps more than any other which encouraged Laue to undertake his remarkable experiments on the diffraction of x-rays by crystals.

Within the last few years Walter has repeated these slit diffraction experiments, making use of the $K\alpha$ line of copper, and has obtained perfectly convincing diffraction effects[14]. Because of the difficulty in determining the width of the slit where the diffraction occurs, it was possible to make from his photographs only a rough estimate of the wave length of the x-rays. But within this rather large probable error the wave length agreed with that determined by crystal spectrometry.

While these slit diffraction experiments were being developed, and long before they were brought to a successful conclusion, Laue and his

[8] E. Wiechert, Sitz. d. Phys-okon Ges. zu Königsberg, 1894.
[9] Haga and Wind, Wied. Ann., *68*, p. 884; 1899.
[10] A. Sommerfeld, Phys. ZS., *2*, p. 59; 1900.
[11] Walter and Pohl, Ann. der Phys., *29*, p. 331; 1909.
[12] P. P. Koch, Ann. der Phys., *38*, p. 507; 1912.
[13] A. Sommerfeld, Ann. der Phys., *38*, p. 473; 1912.
[14] B. Walter, Ann. der Phys., *74*, p. 661; 1924; *75*, Sept. 1924.

collaborators discovered the remarkable fact that crystals act as suitable gratings for diffracting x-rays. You are all acquainted with the history of this discovery. The identity in nature of x-rays and light could no longer be doubted. It gave a tool which enabled the Braggs to determine with a definiteness previously almost unthinkable the manner in which crystals are constructed of their elementary components. By its help Moseley and Siegbahn have studied the spectra of x-rays, we have learned to count one by one the electrons in the different atoms, and we have found out something regarding the arrangement of these electrons. The measurement of x-ray wave lengths thus made possible gave Duane the means of making his precise determination of Planck's radiation constant. By showing the change of wave length when x-rays are scattered, it has helped us to find the quanta of momentum of radiation which had previously been only vaguely suspected. Thus in the two great fields of modern physical inquiry, the structure of matter and the nature of radiation, the discovery of the diffraction of x-rays by crystals has opened the gateway to many new and fruitful paths of investigation. As the Duc de Broglie has remarked, "if the value of a discovery is to be measured by the fruitfulness of its consequences, the work of Laue and his collaborators should be considered as perhaps the most important in modern physics."

These are some of the consequences of extending the optical phenomenon of diffraction into the realm of x-rays.

There is, however, another aspect of the extension of optical diffraction into the x-ray region, which has also led to interesting results. It is the use of ruled diffraction gratings for studies of spectra. By a series of brilliant investigations, Schumann, Lyman and Millikan, using vacuum spectrographs, have pushed the optical spectra by successive stages far into the ultraviolet. Using a concave reflection grating at nearly normal incidence, Millikan and his collaborators[15] found a line, probably belonging to the *L* series of aluminium, of a wave length as short as 136.6A, only a twenty-fifth that of yellow light. Why his spectra stopped here, whether because of failure of his gratings to reflect shorter wave lengths, or because of lack of sensitiveness of the plates, or because his hot sparks gave no rays of shorter wave length, was hard to say.

Röntgen had tried to get x-ray spectra by reflection from a ruled grating, but the task seemed hopeless. How could one get spectra from

[15] Millikan, Bowen, Sawyer, Shallenberger, Proc. Nat. Acad., *7*, p. 289; 1921; Phys. Rev., *23*, p. 1; 1924.

a reflection grating if the grating would not reflect? But when it was found that x-rays could be totally reflected at fine glancing angles, hope for the success of such an experiment was revived. Carrara,[16] working at Pisa, tried one of Rowland's optical gratings, but without success. Fortunately we at Chicago did not know of this failure, and with one of Michelson's gratings ruled specially for the purpose, Doan found that he could get diffraction spectra of the K series radiations from both copper and molybdenum.[17] Fig. 5 shows one of our diffraction spectra, giving several orders of the $K\alpha_1$ line of molybdenum, obtained by reflection at a small glancing angle. This work was quickly

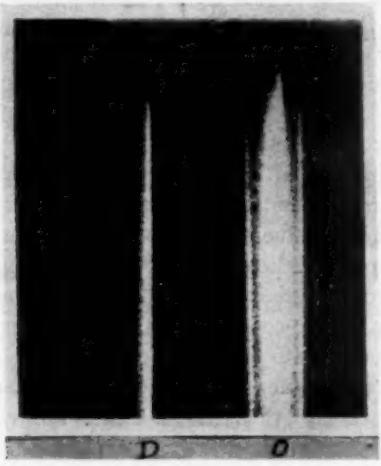

FIG. 5. *Spectrum of the $K\alpha_1$ line of molybdenum, $\lambda = .708A$, from a grating ruled on speculum metal (Compton and Doan). D marks the direct beam, and O the directly reflected beam.*

followed by Thibaud,[18] who photographed a beautiful spectrum of the K series lines of copper from a grating of only a few hundred lines ruled on glass. That x-ray spectra could be obtained from the same type of ruled reflection gratings as those used with light was now established.

The race to complete the spectrum between the extreme ultraviolet of Millikan and the soft x-ray spectra of Siegbahn began again with renewed enthusiasm. It had seemed that the work of Millikan and his coworkers had carried the ultraviolet spectra to as short wave lengths

[16] N. Carrara, N. Cimento, *1*, p. 107; 1924.
[17] A. H. Compton and R. L. Doan, Proc. Nat. Acad., *11*, p. 598; 1925.
[18] J. Thibaud, C. R., Jan. 4, 1926.

as it was possible to go. On the x-ray side, the long wave length limit was placed, theoretically at least, by the spacing of the reflecting layers in the crystal used as a natural grating. De Broglie, W. H. Bragg, Siegbahn and their collaborators were finding suitable crystals of greater and greater spacing, until Thoraeus and Siegbahn,[19] using crystals of palmitic acid, measured the $L\alpha$ line of chromium, with a wave length 21.69A. But there still remained a gap of almost three octaves between these x-rays and the shortest ultraviolet in which, though radiation had been detected by photoelectric methods, no spectral measurements has been made.

Thibaud, working in de Broglie's laboratory at Paris, made a determined effort to extend the limit of the ultraviolet spectrum, using his glass grating at glancing incidence.[20] His spectra, however, stopped at 144A, a little greater than the shortest wave length observed in Millikan's experiments.

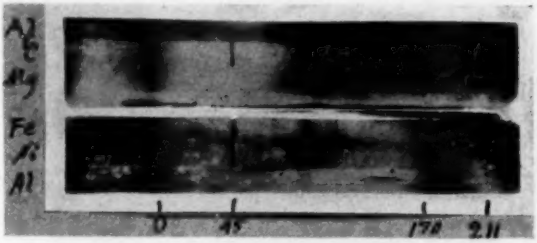

FIG. 6. *Osgood's grating spectra of soft x-rays from Al, C, Mg, Fe and Ni, showing lines from λ=45A to λ=211A. These are the first spectra bridging the gap between the soft x-rays and the ultraviolet.*

But meanwhile Dauvillier, also working with de Broglie, was making rapid strides working from the soft x-ray side of the gap. First,[21] using a grating of palmitic acid, he found the $K\alpha$ line of carbon of wave length 45A. Then[22] using for a grating a crystal of the lead salt of mellissic acid, with the remarkable grating space of 87.5A, he measured a spectrum line of thorium as long as 121A, leaving only a small fraction of an octave between his longest x-ray spectrum lines and Millikan's shortest ultraviolet lines. The credit for filling in the greater part of the remaining gap must thus be given to Dauvillier.

[19] Siegbahn and Thoraeus, Arkiv f M. o F., *19*, p. 1; 1925.
[20] J. Thibaud, J. de Phys. et Rad., *8*, p. 15; 1927.
[21] A. Dauvillier, C. R., *182*, p. 1083; 1926.
[22] A. Dauvillier, J. de Phys. et Rad., *6*, p. 1; Jan. 1927.

The final bridge between the x-ray and the ultraviolet spectra has however been laid by Osgood,[23] a young Scotchman working with me at Chicago. He also used soft x-rays as did Dauvillier, but instead of a crystal grating, he did his experiments with a concave glass grating in a Rowland mounting, but with the rays at glancing incidence. Fig. 6 shows a series of Osgood's spectra. The shortest wave length here shown is the $K\alpha$ line of carbon, 45A, and we see a series of lines up to 211A. An interesting feature of these spectra is an emission band in the aluminium spectrum at about 170A, which is probably in some way associated with the L series spectrum of aluminium. These spectra

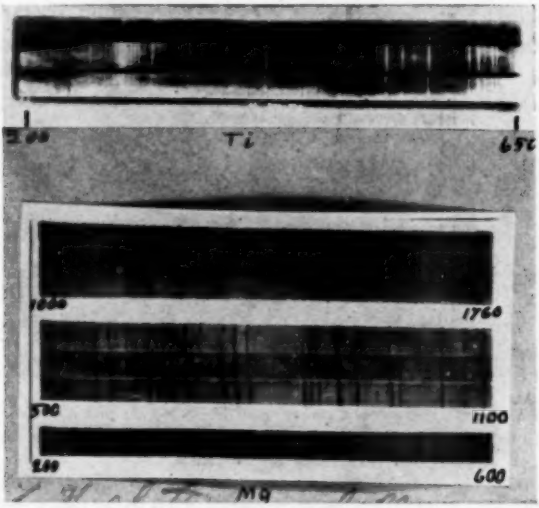

FIG. 7. *Spectra of the extreme ultraviolet, from Mg and Ti, 200A to 1760A (Hoag).*

overlap, on the short wave length side, Dauvillier's crystal measurements, and on the side of the great wave lengths, Millikan's ultraviolet spectra.

In the September number of the Physical Review, Hunt[24] describes similar experiments, using however a plane ruled grating at glancing incidence, in which he has measured lines from 2A down to the carbon line at 45A, thus meeting the shortest of Osgood's measurements. On the other hand, Fig. 7 shows some beautiful spectra of the extreme ultraviolet obtained recently by Dr. Hoag, working with Professor Gale

[23] T. H. Osgood, Nature, *119*, p. 817; June 4, 1927; Phys. Rev., November, 1927.
[24] F. L. Hunt, Phys. Rev., Sept. 1927.

at Chicago, using a concave grating at grazing incidence. These spectra extend from 200A to 1760A, overlapping Osgood's x-ray spectra on the short wave length side, and reaching the ordinary ultraviolet region on the side of the great wave lengths. Thus from the extreme infrared to the region of ordinary x-rays we now have a continuous series of spectra from ruled gratings.

Whatever we may find regarding the nature of x-rays, it would take a bold man indeed to suggest, in light of these experiments, that they differ in nature from ordinary light.

It is too early to predict what may be the consequences of these grating measurements of x-rays. It seems clear, however, that they must lead to a new and more precise knowledge of the absolute wave length of x-rays, and thus to direct determinations of the grating spaces of crystals. This will in turn afford a new means of determining Avogadro's number and the electronic charge, which should be of precision comparable with that of Millikan's oil drops.

THE SCATTERING OF X-RAYS AND LIGHT

The phenomena that we have been considering are ones in which the laws which have been found to hold in the optical region apply equally well in the x-ray region. This is not the case, however, for all optical phenomena.

The theory of the diffuse scattering of light by turbid media has been examined by Drude, Lord Rayleigh, Raman and others, and an essentially similar theory of the diffuse scattering of x-rays has been developed by Thomson, Debye and others. Two important consequences of these theories are, (1) that the scattered radiation shall be of the same wave length as the primary rays, and (2) that the rays scattered at 90 degrees with the primary rays shall be plane polarized. The experimental tests of these two predictions have led to interesting results.

A series of experiments performed during the last few years[25] have shown that secondary x-rays are of greater wave length than the primary rays which produce them. This work is too well known to require description. On the other hand, careful experiments to find a similar increase in wave length in light diffusely scattered by a turbid medium have failed to show any such effect.[26] An examination of the spectrum

[25] For an account of this work, see e.g. the writer's "X-Rays and Electrons," Chap. IX, Van Nostrand, 1926.

[26] E. g., P. A. Ross, Proc. Nat. Acad., *9*, p. 246; 1923.

of the secondary x-rays shows that the primary beam has been split into two parts, as shown in Fig. 8, one of the same wave length and the other of increased wave length. When different primary wave lengths are used, we find always the same difference in wave length between these two components; but the relative intensity of the two components changes. For the longer wave lengths the unmodified ray has the greater energy, while for the shorter wave lengths the modified ray is predominant. In fact when hard γ-rays are employed, it is not possible to find any radiation of the original wave length.

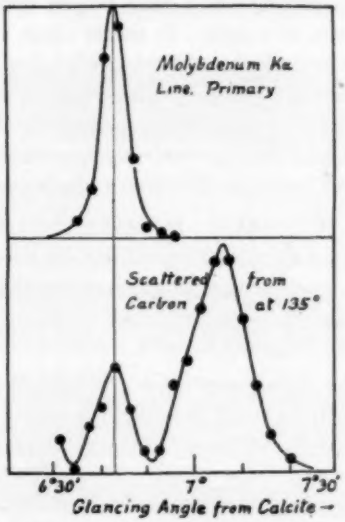

FIG. 8. *A typical spectrum of scattered x-rays, showing the splitting of the primary ray into a modified and an unmodified ray.*

Thus in the wave length of secondary radiation we have a gradually increasing departure from the classical electron theory of scattering as we go from the optical region to the region of x-rays and γ-rays.

The question arises, are these secondary x-rays of increased wavelength to be classed as scattered x-rays or as fluorescent rays? An important fact bearing on this point is the intensity of the secondary rays. From the theories of Thomson, Debye and others it is possible to calculate the absolute intensity of the scattered rays. It is found that this calculated intensity agrees very nearly with the total intensity of the modified and unmodified rays, but that in many cases the observed intensity of the unmodified ray taken alone is very small compared with

the calculated intensity. If the electron theory of the intensity of scattering is even approximately correct, we must thus include the modified with the unmodified rays as scattered rays.

Information regarding the origin of these secondary rays is also given by a study of their state of polarization. We have called attention to the fact that the electron theory demands that the x-rays scattered at 90 degrees should be completely plane polarized. If the rays of increased wave-length are fluorescent, however, we should not expect them to be strongly polarized. You will remember the experiments performed by Barkla[27] some twenty years ago in which he observed strong polarization in x-rays scattered at right angles. It was this experiment which gave us our first strong evidence of the similar character of x-rays and light. But in this work the polarization was far from complete. In fact the intensity of the secondary rays at 90 degrees dropped only to one third of its maximum value, whereas for complete polarization it should have fallen to zero. It might have seemed that the remaining third was due to really unpolarized rays of a fluorescent type.

The fact that no such unpolarized rays exist was established by repeating Barkla's experiment[28] with scattering blocks of different

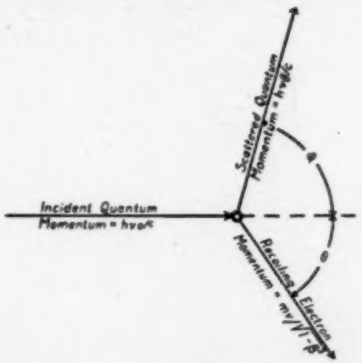

FIG. 9. *An x-ray photon is deflected through an angle ϕ by an electron, which in turn recoils at an angle θ, taking up a part of the energy of the photon.*

sizes. When very small blocks were used, we found that the polarization was nearly complete. The lack of complete polarization in Barkla's experiments was due chiefly to the multiple scattering of the x-rays in the large blocks that he used to scatter the x-rays. It would seem that

[27] C. G. Barkla, Proc. Roy. Soc. A., 77, p. 247; 1906.
[28] A. H. Compton and C. F. Hagenow, J.O.S.A. and R.S.I., 8, p. 487; 1924.

the only explanation of the complete polarization of the secondary rays is that they consist wholly of scattered rays.

According to the classical theory, an electromagnetic wave is scattered when it sets the electrons which it traverses into forced oscillations, and these oscillating elecrons reradiate the energy which they receive. In order to account for the change in wave-length of the scattered rays, however, we have had to adopt a wholly different picture of the scattering process—that shown in Fig. 9. Here we do not think of the x-rays as waves, but as light corpuscles, quanta, or, as we may call them, photons. Moreover, there is nothing here of the forced oscillation pictured on the classical view, but a sort of elastic collision, in which the energy and momentum are conserved.

This new picture of the scattering process leads at once to three consequences that can be tested by experiment. There is a change of wave-length

$$\delta\lambda = \frac{h}{mc}(1-\cos\phi), \tag{1}$$

which accounts for the modified line in the spectra of scattered x-rays. Experiment has shown that this formula is correct within the precision of our knowledge of h, m and c. The electron which recoils from the scattered x-ray should have the kinetic energy,

$$E_{\text{kin}} = h\nu \cdot \frac{h\nu}{mc^2}\cos^2\theta, \tag{2}$$

approximately. When this theory was first proposed, no electrons of this type were known; but they were discovered by Wilson[29] and Bothe[30] within a few months after their prediction. Now we know that the number, energy and spatial distribution of these recoil electrons are in accord with the predictions of the photon theory. Finally, whenever a photon is deflected at an angle ϕ, the electron should recoil at an angle θ given by the relation,

$$\cot\tfrac{1}{2}\phi = \tan\theta, \tag{3}$$

approximately.

This relation we have tested[31] using the apparatus shown diagrammatically in Fig. 10. A narrow beam of x-rays enters a Wilson expansion

[29] C. T. R. Wilson, Proc. Roy. Soc., *109*, p. 1; 1923.
[30] W. Bothe, ZS. f. Phys., *16*, p. 319; 1923; *20*, p. 237; 1923.
[31] A. H. Compton and A. W. Simon, Phys. Rev., *26*, p. 289; 1925.

chamber. Here it produces a recoil electron. If the photon theory is correct, associated with this recoil electron, a photon is scattered in the direction ϕ. If it should happen to eject a β-ray, the origin of this

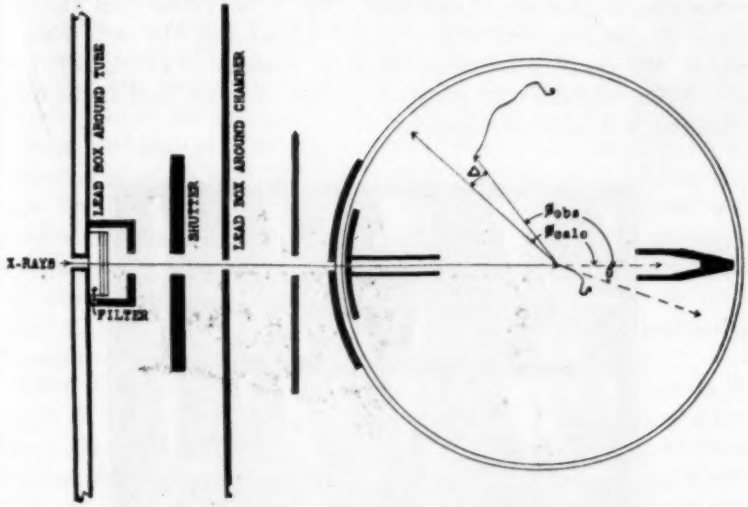

FIG. 10. *An electron recoiling at an angle θ should be associated with a photon deflected through an angle ϕ.*

β-ray tells the direction in which the photon was scattered. Fig. 11 shows a typical photograph of the process. A measurement of the angle θ at which the recoil electron on this plate is ejected and the angle ϕ of the origin of the secondary β-particle, shows close agreement with the photon formula. This experiment is of especial significance, since it shows that for each recoil electron there is a scattered photon, and that the energy and momentum of the system photon plus electron are conserved in the scattering process.

The evidence for the existence of directed quanta of radiation afforded by this experiment is very direct. The experiment shows that associated with each recoil electron there is scattered x-ray energy enough to produce a secondary beta ray, and that this energy proceeds in a direction determined at the moment of ejection of the recoil electron. Unless the experiment is subject to improbably large experimental errors, therefore, *the scattered x-rays proceed in the form of photons.*

Thus we see that as a study of the scattering of radiation is extended into the very high frequences of x-rays, the manner of scattering

changes. For the lower frequencies the phenomena could be accounted for in terms of waves. For these higher frequencies we can find no interpretation of the scattering except in terms of the deflection of corpuscles or photons of radiation. Yet it is certain that the two types of radiation, light and x-rays, are essentially the same kind of thing. We are thus confronted with the dilemma of having before us convincing evidence that radiation consists of waves, and at the same time that it consists of corpuscles.

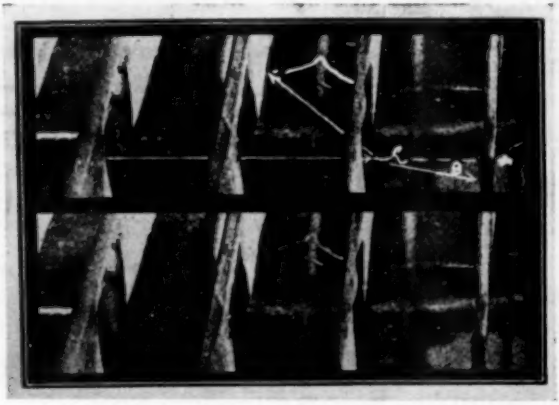

FIG. 11. *Photograph showing recoil electron and associated secondary β-ray. The upper photograph is retouched.*

It would seem that this dilemma is being solved by the new wave mechanics. De Broglie[32] has assumed that associated with every particle of matter in motion there is a wave whose wave length is given by the relation,

$$mv = h/\lambda,$$

where mv is the momentum of the particle. A very similar assumption was made at about the same time by Duane,[33] to account for the diffraction of x-ray photons. As applied to the motion of electrons, Schrödinger has shown the great power of this conception in studying atomic structure.[34] It now seems, through the efforts of Heisenberg, Bohr and others, that this conception of the relation between corpuscles

[32] L. de Broglie, Thèse, Paris, 1924.

[33] W. Duane, Proc. Nat. Acad., 1924.

[34] E. Shrödinger, Ann. der Phys., *79*, pp. 361, 489, 734; *80*, 437; *81*, 109; 1926; Phys. Rev., *28*, p. 1051; 1926.

and waves is capable of giving us a unified view of the diffraction and interference of light, and at the same time of its diffuse scattering and of the photoelectric effect. It would however take too long to describe these new developments in detail.

We have thus seen how the essentially optical properties of radiation have been recognized and studied in the realm of x-rays. A study of the refraction and specular reflection of x-rays has given an important confirmation of the electron theory of dispersion, and has enabled us to count with high precision the number of electrons in the atom. The diffraction of x-rays by crystals has given wonderfully exact information regarding the structure of crystals, and has greatly extended our knowledge of spectra. When x-rays were diffracted by ruled gratings, it made possible the study of the complete spectrum from the longest to the shortest waves. In the diffuse scattering of radiation, we have found a gradual change from the scattering of waves to the scattering of corpuscles.

Thus by a study of x-rays as a branch of optics we have found in x-rays all of the well known wave characteristics of light, but we have found also that we must consider these rays as moving in directed quanta. It is these changes in the laws of optics when extended to the realm of x-rays which have been in large measure responsible for the recent revision of our ideas regarding the nature of the atom and of radiation.

UNIVERSITY OF CHICAGO,
CHICAGO, ILLINOIS,
OCTOBER 10, 1927.

VECTORIAL TREATMENT OF REFRACTION OF SKEW RAYS BY A PRISM

By Ludwik Silberstein*

The usual trigonometric derivation of the formulas for the refraction in a prism is, in the special case of rays contained in the principal section (perpendicular to the edge), simple enough. Not so, however, in the more general case of oblique or skew incident rays, when the deduction commonly given in optical treatises spreads over several pages.[1] It is in this more intricate case of skew rays that the vectorial treatment offers the advantage of considerable simplification.

Let the unit vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the normals of the first and the second face of the prism, say both drawn outward, so that, ϵ being the refracting angle of the prism and $\boldsymbol{e}$ a unit vector along its edge, $\boldsymbol{n}_1\boldsymbol{n}_2 = -\cos\epsilon$, $V\boldsymbol{n}_1\boldsymbol{n}_2 = \boldsymbol{e}\sin\epsilon$. This, together with the refractive index μ, completely characterizes the prism. Let the unit vector $\boldsymbol{r}$ represent any given incident ray. Then the twice refracted and finally emergent ray $\boldsymbol{r}'$, another unit vector, is

$$\boldsymbol{r}' = \boldsymbol{r} + g_1\boldsymbol{n}_1 + g_2\boldsymbol{n}_2 \tag{1}$$

where g_1, g_2 are scalar coefficients; in obvious symbols,

$$g_1 = \cos i_1 - \mu\cos i_1' = \cos i_1 - (\mu^2 - \sin^2 i_1)^{1/2},$$

and similarly for g_2. The latter scalar, moreover, can be readily eliminated, viz. by multiplying (1) vectorially by $\boldsymbol{n}_2$, which gives

$$V(\boldsymbol{r}' - \boldsymbol{r})\boldsymbol{n}_2 = g_1\sin\epsilon\,\boldsymbol{e}. \tag{2}$$

These formulas, of which the first follows directly from the refraction law, were given in my little book on vectorial treatment of optics[2] without, however, being expanded any further. As a matter of fact, these formulas already determine completely the emergent ray in terms of the incidence data represented by the vector $\boldsymbol{r}$. But it is manifestly desirable to express the vectorial relation in terms of some appropriate angular coordinates of the rays.

* Communication No. 333 from the Kodak Research Laboratories. Paper read at the annual meeting of the Optical Society of America at Schenectady, October 22, 1927.

[1] *Cf.* for instance, M. v. Rohr's Formation of Images, *etc.*, p. 433, London, 1920.

[2] Simplified Method of Tracing Rays, *etc.* Longmans, p. 17; 1918. In formula (17), 1/sinϵ should read sinϵ.

Thus, let the incident ray be fixed by its inclination θ to the prism edge, so that $\cos\theta = \boldsymbol{re}$, and by the angle ϕ between the plane $\boldsymbol{r}$, $\boldsymbol{e}$ and the first face of the prism.[3] Let θ', ϕ' be the corresponding angles for

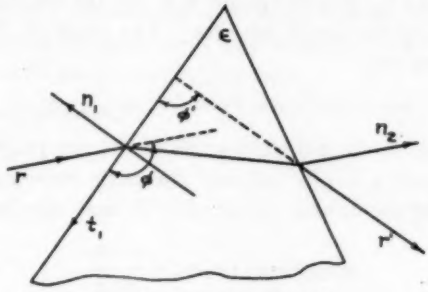

FIG. 1. *Symbols explained.*

the emergent ray. In symbols, and introducing the unit vector $\boldsymbol{t}_1 = V\boldsymbol{n}_1\boldsymbol{e}$ in the plane of the first prism face normal to the edge,

$$\left.\begin{aligned} \boldsymbol{r} &= \boldsymbol{e}\cos\theta + \sin\theta(\boldsymbol{t}_1\cos\phi - \boldsymbol{n}_1\sin\phi) \\ \boldsymbol{r}' &= \boldsymbol{e}\cos\theta' + \sin\theta'(\boldsymbol{t}_1\cos\phi' - \boldsymbol{n}_1\sin\phi'). \end{aligned}\right\} \tag{3}$$

Then the problem is: given θ,ϕ, find θ',ϕ'. Now, since the edge is perpendicular to both normals, equation (1) gives $\boldsymbol{r}'\boldsymbol{e} = \boldsymbol{re}$ or $\cos\theta' = \cos\theta$,

$$\theta' = \theta. \tag{4}$$

The inclination to the edge and thus also to the principal plane remains unaltered,—a known property which here, however, follows from (1) on inspection. Thus it remains to determine ϕ'.

Now, by (2), (3), and (4),

$$(\cos\phi' - \cos\phi)V\boldsymbol{t}_1\boldsymbol{n}_2 - (\sin\phi' - \sin\phi)V\boldsymbol{n}_1\boldsymbol{n}_2 = \frac{g_1\sin\epsilon}{\sin\theta}\boldsymbol{e}.$$

But $V\boldsymbol{t}_1\boldsymbol{n}_2 = -\boldsymbol{e}\cos\epsilon$, $V\boldsymbol{n}_1\boldsymbol{n}_2 = \boldsymbol{e}\sin\epsilon$.

Hence

$$\cos(\phi' - \epsilon) = \cos(\phi - \epsilon) - \frac{g_1\sin\epsilon}{\sin\theta}, \tag{5}$$

which is the required formula. The value of g_1 is, as before,

$$g_1 = \cos i_1 - (\mu^2 - 1 + \cos^2 i_1)^{1/2},$$

[3] Thus $\phi - \pi/2$ will be the incidence angle of the ray projected upon the principal plane.

where, in terms of the data θ,ϕ, by (3),

$$\cos i_1 = -\boldsymbol{r}\boldsymbol{n}_1 = \sin\theta\sin\phi.$$

Equations (4) and (5) give the angles θ',ϕ' for the emergent ray explicitly in terms of the incidence data θ,ϕ. The resultant deviation $\tilde{\omega}$ follows at once from (3),

$$\cos\tilde{\omega} = \boldsymbol{r}\boldsymbol{r}' = \cos^2\theta + \sin^2\theta\cos(\phi'-\phi). \tag{6}$$

Since $\phi-\pi/2=\psi_1$ is the incidence angle of the ray projected upon the principal plane and $\pi/2-(\phi'-\epsilon)=\psi_2'$ the angle between the projected emergent ray and the normal $\boldsymbol{n}_2$, formula (5) may also be written

$$\sin\psi_2' = \sin(\epsilon-\psi_1) - \frac{g_1\sin\epsilon}{\sin\theta}. \tag{5'}$$

This settles the general case.

In particular, if the rays are contained in the principal plane $\theta=\pi/2, \psi_1=i_1, \psi_2'=i'_2$, and formula (5') reduces to

$$\sin i_2' = \sin(\epsilon-i_1) - g_1\sin\epsilon. \tag{5_0}$$

The familiar treatment of this case consists in writing down the set of equations

$$\sin i_1' = \frac{1}{\mu}\sin i_1, \quad i_2 = \epsilon - i_1', \quad \sin i_2' = \mu\sin i_2.$$

From these we find

$$\sin i_2' = \mu\sin\epsilon\cos i_1' - \sin i_1\cos\epsilon = (\cos i_1 - g_1)\sin\epsilon - \sin i_1\cos\epsilon,$$

thus verifying (5_0).

Returning to the general case let us still express in terms of θ,ϕ the condition $\sin^2 i'_2<1$ necessary and sufficient for the rays to emerge; there being total reflection at the second face of the prism, if this condition is not satisfied If $\boldsymbol{r}_2$ be the ray incident upon the second face, we have $\mu\boldsymbol{r}_2\equiv\mu\boldsymbol{r}'_1=\boldsymbol{r}+g_1\boldsymbol{n}_1$, whence $\mu\cos i_2=\mu\boldsymbol{r}_2\boldsymbol{n}_2=\boldsymbol{r}\boldsymbol{n}_2-g_1\cos\epsilon$. Now, by the first of (3) and since $\boldsymbol{t}_1\boldsymbol{n}_2=-\sin\epsilon$, $\boldsymbol{n}_1\boldsymbol{n}_2=-\cos\epsilon$,

$$\boldsymbol{r}\boldsymbol{n}_2 = \sin\theta\sin(\phi-\epsilon).$$

Thus, and since $\sin^2 i'_2=\mu^2(1-\cos^2 i_2)$, the required condition is

$$[\sin\theta\sin(\phi-\epsilon) - g_1\cos\epsilon]^2 > \mu^2-1$$

or, ultimately,

$$[\sin\epsilon\sin\theta\cos\phi - \cos\epsilon(\mu^2-1+\sin^2\phi\sin^2\theta)^{1/2}]^2 > \mu^2-1. \tag{7}$$

The discussion of this condition for any given incident ray offers no difficulties. Notice, for instance, that an incident ray parallel to the edge, $r=e$ ($\theta=0$), will always be totally reflected at the second face of the prism, for then $[\ ]^2=\cos^2\epsilon(\mu^2-1)$, while for any grazing incidence, $\phi=\pi$, the condition becomes $[\sin\epsilon\sin\theta+\cos\epsilon\ (\mu^2-1)^{1/2}]^2>\mu^2-1$, and if θ be large enough, such rays may emerge.

KODAK RESEARCH LABORATORIES,
ROCHESTER, N. Y.,
OCTOBER, 1927.

NOTE ON AN APPARENT NON-PROPORTIONALITY OF PHOTOELECTRIC CURRENTS

By B. Kurrelmeyer*

While measuring the absorption of a crystal with a photoelectric cell and an electrometer the author's attention was drawn to the existence of an apparent lack of proportionality between light intensity and photocurrent. The deviations were of the order of five per cent in the extreme cases, and were much larger than the errors in the readings. They were also, however, larger than the deviations which experience would lead one to expect in photocells of the type used. It was found that the deviations were due to a "feedback" effect; that is, that the potential difference of the charges accumulating on the electrometer reduced the potential difference across the photocell by an amount sufficient to produce the observed lack of proportionality. Since others may encounter this difficulty, it is proposed to give a short account of the circumstances.

The cell was a gas-filled potassium hydride cell having the usual current-voltage characteristic, (current increasing rapidly with voltage, due to ionization). The photocurrent was observed by measuring the rate of charge of a string electrometer, (capacity, with leads and photocell, 15 micromicrofarads). Potentials up to 200 volts were applied to the cell, and the currents measured charged the electrometer up to several volts in five or ten seconds, corresponding to currents of the order of 10^{-11} ampere, no leak resistance being used.

The key to the solution of the difficulty was furnished by the observation that the charges measured were not proportional to the *time* of illumination any more than to the intensity, but that if the product of time by intensity were kept constant, the total charge measured was always the same. This, together with the known dependence of the photocurrent on the cell voltage, led to the quantitative calculation of the effect. The procedure is obvious. The result is found to be

$$R_c = R\left[1+\frac{\Delta R_1}{2}+\cdots+\frac{1}{n}(\Delta R_1)^{n-1}\right]$$

where R_c is the correct electrometer reading, R the actual reading, ΔR_1 the change in R produced by a unit change in the cell voltage; in

* National Research Fellow.

general only the first correction term need be used. ΔR_1 may be determined in terms of R for any given cell voltage at the beginning of the measurements, so that the R_c for any given R can be seen at a glance.

The check on the calculation consisted in measuring R for several different times of illumination, say, 5, 10, and 15 seconds, and finding the corresponding R_c's. An actual example is given. The agreement is within the limits of the experimental errors.

R	R_c	Ratio of R_c's
1.29	1.29	1
2.52	2.58	2.00
3.74	3.90	3.02

JEFFERSON PHYSICAL LABORATORY,
HARVARD UNIVERSITY,
JUNE, 1927.

ON THE CARDINAL POINTS OF A THICK LENS FOR RAYS WHICH HAVE SUFFERED TWO INTERNAL REFLECTIONS

By James P. C. Southall

A brief paper by Dr. G. F. C. Searle with the same title as above appeared recently in the published *Proceedings of the Optical Convention*, 1926 (London), Part II, pp. 825–829, which contained the whole theory of this problem. The problem itself is doubtless not one of much practical importance, and yet because it illustrates in a simple way some peculiar types of optical systems and is particularly instructive to the student of geometrical optics on this account, it seems worth while to study it here a little more in detail and especially to approach it in a somewhat different fashion chiefly for the purpose of showing how the ordinary formulas and methods may easily be applied to an optical system of this type in which the light has suffered one or more internal reflections. The particular method which is here employed is an exceedingly convenient one for which the writer is originally indebted to an elegant paper published some years ago by Mr. T. Smith*, and an outline of which may be found in the revised edition of Southall's *Mirrors, Prisms, and Lenses* (The Macmillan Company, New York, 1923), pages 608–611.

It will be assumed for simplicity that the lens is surrounded by air, being made of glass whose index of refraction will be denoted by n. The vertices of the lens where the optical axis crosses the two surfaces will be designated by A_1 and A_2, and the centers of curvature of the spherical faces by C_1 and C_2; the radii r_1 and r_2 being defined as the steps from A_1 to C_1 and from A_2 to C_2, respectively, and the thickness of the lens (d) as the step from the first vertex A_1 to the second vertex A_2; that is,

$$r_1 = A_1C_1, \quad r_2 = A_2C_2, \quad nc = d = A_1A_2,$$

where c denotes therefore the so-called reduced thickness. The curvatures of the two surfaces will be denoted by

$$R_1 = 1/r_1 \text{ and } R_2 = 1/r_2.$$

The direction along the axis pursued by the *incident* light (which in a diagram is conveniently represented by a horizontal line with an

* T. Smith: Note on optical imagery. Proc. Phys. Soc. Lond., *25*, pp. 239–44; 1912–3.

arrow-head pointing towards the right) is reckoned here as the *positive direction*. Thus, for example, the vertices and centers of curvature of the two surfaces of a double concave lens being ranged along the axis in the sequence C_1, A_1, A_2, C_2, from left to right if the incident light has this direction, the radius of the first surface is negative in sign and that of the second surface positive.

In the problem here under consideration the rays undergo four successive deflections, namely, refraction on entering the lens by the first face, two reflections, first at the first face and then at the second, and, finally, refraction on leaving the lens by the second face. Whenever a reflection occurs, it involves a change of sign of both the index of refraction (n) and the thickness of the lens (d); consequently, the reduced thickness $c=d/n$ remains unchanged in both sign and magnitude throughout all these successive performances. It is important to bear this in mind and to note that the magnitude denoted by c is a *positive constant* under all circumstances. It is true that cases do arise when c is apparently negative; but in any such instance it will be found that this is in reality no exception to the above statement and that for some reason or other the direction of the incident light has been reversed, since c must be measured always in that direction.

The refracting powers of the first and second surfaces of the lens will be denoted by F_1 and F_4, and the reflecting powers by F_2 and F_3, respectively; or, to be specific, we shall put

$$F_1=(n-1)R_1, \quad F_2=-2nR_2, \quad F_3=2nR_1, \quad F_4=-(n-1)R_2,$$

Knowing the numerical values of n, R_1, R_2 and c by which the nature, form and dimensions of the lens are specified, we proceed now to perform two sets of operations as follows, in the order indicated:

X-calculation

$$\begin{aligned} X_1&=1, & F_{11}&=F_1,\\ X_2&=X_1-c\cdot F_{11}, & F_{12}&=F_{11}+X_2\cdot F_2,\\ X_3&=X_2-c\cdot F_{12}, & F_{13}&=F_{12}+X_3\cdot F_3,\\ X_4&=X_3-c\cdot F_{13}, & F_{14}&=F_{13}+F_4,\\ & & F&=F_{14}. \end{aligned}$$

Y-calculation

$$\begin{aligned} Y_4&=1, & F_{44}&=F_4,\\ Y_3&=Y_4-c\cdot F_{44}, & F_{34}&=F_{44}+Y_3\cdot F_3,\\ Y_2&=Y_3-c\cdot F_{34}, & F_{24}&=F_{34}+Y_2\cdot F_2,\\ Y_1&=Y_2-c\cdot F_{24}, & F_{14}&=F_{24}+Y_1\cdot F_1,\\ & & F&=F_{14}. \end{aligned}$$

Thus each of the two calculations leads to the same final result, namely, the value of the refracting power F of the optical system as a whole, and therefore one serves to check the other. At the same time the two processes enable us to ascertain also the positions of the cardinal points of the system, namely, the pair of focal points (F, F′) and the pair of so-called principal points (H, H′); for having found the values of X_4 and F by the first calculation, and the values of Y_1 and F by the second, we can locate the positions of the cardinal points of the system with respect to the vertices of the lens by means of the following relations:

$$A_1F=-\frac{Y_1}{F}, \qquad A_1H=\frac{1-Y_1}{F},$$

$$A_2F'=\frac{X_4}{F}, \qquad A_2H'=\frac{X_4-1}{F}.$$

A simple numerical illustration will make the process clear and at the same time will give an opportunity of showing how the arithmetical results may be conveniently tabulated according to T. Smith's original method. For this purpose let us consider the case when the optical system is composed of a *concentric meniscus lens*, made of glass of index 1.5, and in the following dimensions: Thickness, $d=1/120$; curvatures, $R_1=30$, $R_2=40$. With these numbers the calculation, as exhibited in the subjoined table, can be easily followed step by step:

Concentric Meniscus

$n=1.5$, $F_1=15$, $F_2=-120$, $F_3=90$, $F_4=-20$, $c=1/180$.

X-calculation					Y-calculation				
k	X_k	$F_k \cdot X_k$	F_{1k}	$-c \cdot F_{1k}$	k	Y_k	$F_k \cdot Y_k$	F_{k4}	$-c \cdot F_{k4}$
1	1	15	15	$-\frac{1}{12}$	4	1	− 20	−20	$\frac{1}{9}$
2	$\frac{11}{12}$	−110	−95	$\frac{19}{36}$	3	$\frac{10}{9}$	100	80	$-\frac{4}{9}$
3	$\frac{13}{9}$	130	35	$-\frac{7}{36}$	2	$\frac{2}{3}$	−80	0	0
4	$\frac{5}{4}$	− 25	10		1	$\frac{2}{3}$	10	10	

In each of the two divisions marked X and Y, the first entry (line 1, column 2) is always unity. Thereafter each number in this column is

the algebraic sum of the number above it and the number in the last column of the preceding line. The two numbers on the first line in columns 3 and 4 are always identical; thereafter the entry to be placed in column 4 is the algebraic sum of the amounts on the same line in column 3 and on the preceding line in column 4. The final results of the two calculations, as already stated, are invariably exactly the same. In the above illustration we find:

$$F=10, \quad X_4=\frac{5}{4}, \quad Y_1=\frac{2}{3},$$

and hence

$$A_1F=-\frac{1}{15}, \quad A_1H=\frac{1}{30}, \quad A_2F'=\frac{1}{8}, \quad A_2H'=\frac{1}{40},$$

which may be checked with each other, by noting that in any optical system surrounded by the same medium on both sides

$$FH=FA_1+A_1H=H'A_2+A_2F'=\frac{1}{F}=f,$$

where f denotes therefore the focal length. If the unit of length is one meter, the curvatures and refracting powers will all be given in dioptries. Accordingly, if in this example, $F=10$ dptr, and if the linear dimensions are finally reduced to centimeters, the results will be expressed as follows:

$$A_1A_2=0.83, \qquad A_2C_1=A_2C_2=A_2H=A_2H'=2.50,$$
$$A_1F=-6.67, \qquad A_2F'=12.50;$$

so that in this particular case we see that the four points H, H′, C_1, C_2 all coincide 2.5 cm beyond the vertex of the second surface of the lens.

The X- and Y-calculations may be combined into three fundamental formulas applicable to any system of the type here under consideration, namely:

$$X_4=1-c(3F_1+2F_2+F_3)$$
$$+c^2(2F_1F_2+F_2F_3+2F_3F_1)-c^3F_1F_2F_3;$$

$$Y_1=1-c(3F_4+2F_3+F_2)$$
$$+c^2(2F_4F_3+F_3F_2+2F_2F_4)-c^3F_4F_3F_2;$$

$$F=F_1+F_2+F_3+F_4 \\ -c(F_1F_2+2F_1F_3+F_2F_3+3F_1F_4+2F_2F_4+F_3F_4) \\ +c^2(F_1F_2F_3+2F_1F_2F_4+2F_1F_3F_4+F_2F_3F_4) \\ -c^3F_1F_2F_3F_4.$$

It requires considerable study to determine the shape and dimensions of a lens for which the optical system will have a prescribed refracting power, and many of the solutions are found to be quite impractical. The problem will therefore now be examined more particularly in order to enable the investigator to obtain some insight into the conditions that are involved. However, in order to pursue the analytical investigation without having to deal with too complicated expressions, it is found to be advantageous to assume some definite numerical value of the index of refraction of the lens; and hereafter it will be taken for granted that

$$n=1.5,$$

which is a convenient value for mathematical purposes and not likely to be so different from the value in any actual case that the results cannot be utilized with advantage, even if they are not strictly applicable. Accordingly, we may write now:

$$d=1.5c, \quad F_1=0.5R_1, \quad F_2=-3R_2, \\ F_3=3R_1, \quad F_4=-0.5R_2, \quad (n=1.5).$$

At the same time let us introduce three new symbols defined as follows:

$$\epsilon=\frac{R_2}{R_1}, \quad \phi=\frac{F}{R_1}, \quad x=c\cdot R_1,$$

which are therefore connected by the following relations:

$$\phi\cdot x=c\cdot F, \quad \phi\cdot R_2=\epsilon\cdot F, \quad \epsilon x=c\cdot R_2.$$

Accordingly, eliminating F_1, F_2, F_3, F_4, F and c from the fundamental equations above by expressing these magnitudes in terms of ϵ, ϕ and x, we obtain:

$$X_4=1+\frac{3(4\epsilon-3)}{2}x+3(1-4\epsilon)x^2+\frac{9\epsilon}{2}x^3,$$

$$Y_1=1-\frac{3(4-3\epsilon)}{2}x+3\epsilon(\epsilon-4)x^2-\frac{9\epsilon^2}{2}x^3 \\ =X_4-\frac{3(1+\epsilon)x\{1+2(1-\epsilon)x+3\epsilon x^2\}}{2},$$

and a quadratic equation of the form

$$A\epsilon^2+B\epsilon+C=0,$$

where the coefficients A and B are functions of x, and the absolute term C is a function of x and ϕ, namely:

$$A=3x(3x-2)(x-2),\quad B=24x^2-51x+14,\quad C=2(6x-7+2\phi).$$

If

$$\Delta=B^2-4AC,$$

the roots of this equation will be

$$\epsilon=\frac{-B\pm(\Delta)^{1/2}}{2A}.$$

The form of the lens will depend on the value of ϵ which is itself dependent on two arbitrary independent variables ϕ and x; and the conditions may be thus formulated:

1. If ϵ is negative, the signs of R_1 and R_2 are unlike. Accordingly, the lens will be *double convex* ($R_1>0$) or *double concave* ($R_1<0$). In particular, when $\epsilon=-1$ (or $A-B+C=0$), the lens will be *equi-biconvex* ($R_1>0$) or *equi-biconcave* ($R_1<0$).

2. If $\epsilon=0$, in which case $C=0$ also (or $2\phi=7-6x$), the lens will be *planoconvex* ($R_1>0$, $R_2=0$) or *planoconcave* ($R_1<0$, $R_2=0$).

3. Lastly, when ϵ is positive, the curvatures R_1 and R_2 will have like signs and the lens will have the form of a *meniscus*. If the value of ϵ lies between 0 and $+1$, the *meniscus* will be crescent-shaped or *convex* when $R_1>0$, and canoe-shaped or *concave* when $R_1<0$; whereas if $\epsilon>1$, the meniscus will be *concave* when $R_1>0$ and *convex* when $R_1<0$. When $\epsilon=1$ (or $A+B+C=0$), the lens will be a "meniscus of zero curvature" ($R_2-R_1=0$).

Accordingly, taking account also of the relations connecting ϕ, x, c and F, we may classify these optical systems in two main groups depending simply on whether the first surface of the lens is convex or concave, as shown in the subjoined table:

	I R_1 and x both positive, the sign of ϕ being therefore the same as that of the refracting power F.	II R_1 and x both negative, the sign of ϕ being therefore opposite to that of the refracting power F.
$\epsilon<0$	Double convex lens	Double concave lens
$\epsilon=0$	Planoconvex lens ($R_2=0$)	Planoconcave lens ($R_2=0$)
$0<\epsilon<1$	Convex meniscus ($R_2<R_1$)	Concave meniscus ($R_2>R_1$)
$\epsilon>1$	Concave meniscus ($R_2>R_1$)	Convex meniscus ($R_2<R_1$)

The above classification includes all possible combinations inasmuch as the refracting power of the system is not altered by reversing the attitude of the lens towards the incident light. Moreover, it will be sufficient in each case to make the calculations on the assumption that the refracting power of the system is equal to $1(F=+1$ for positive systems and $F=-1$ for negative systems), because, having once located the positions of the cardinal points for unit value of F, we can immediately obtain all the corresponding data for any other value of F simply by dividing all the linear dimensions by this new number (which, of course, is equivalent to multiplying the curvatures, etc., by this same factor). In this connection it may be well to remark that in case $F=+1$, we must put $\phi=r_1=1/R_1$ and $\phi x=c$, whereas, when $F=-1$, then $\phi=-r_1$ and $\phi x=-c$.

At this stage perhaps several actual numerical illustrations will serve better than anything else to make matters clear. Since both x and ϕ are arbitrary magnitudes, we may begin by assuming some convenient numerical value of x, say, $x=0.1$, in which case the coefficients A and B of the quadratic equation in ϵ will have the values $A=0.969$, $B=9.140$, the absolute term being a function of ϕ, namely, $C=4(\phi-3.2)$. In this instance x is positive and hence the optical system belongs to Group I above. In addition, let us assume next that $\phi=2$; which means that the refracting power of the system must be positive also. Accordingly, $F=+1$, $c=\phi x=0.2$, that is $d=0.3(n=1.5)$. Moreover $C=-4.8$, and therefore $(\Delta)^{1/2}=10.10665$. Thus one of the roots of the quadratic equation proves to be positive and the other negative, the lens being therefore a meniscus lens when $\epsilon>0$ and a double convex lens when $\epsilon<0$. Here let us pursue the former case for which $\epsilon=0.49879(x=0.1, \phi=2)$, which, being less than unity, determines the

form of the lens as that of a *convex meniscus* for which $R_1 = 1/\phi = 0.5$ and $R_2 = \epsilon R_1 = \epsilon/\phi = 0.24940$. Consequently, since n = 1.5, we have here:

$$F_1 = 0.25000, \quad F_2 = -0.74820, \quad F_3 = 1.50000, \quad F_4 = -0.12470.$$

The rest of the calculation is exhibited below, according to the method already explained:

Lens: $R_1 = 0.50000$, $R_2 = 0.24940$, $d = 0.30000$, n = 1.5

X-calculation					*Y*-calculation				
k	X_k	$F_k \cdot X_k$	F_{1k}	$-c \cdot F_{1k}$	k	Y_k	$F_k \cdot Y_k$	F_{4k}	$-c \cdot F_{4k}$
1	1.00000	0.25000	0.25000	−0.05000	4	1.00000	−0.12470	−0.12470	0.02494
2	0.95000	−0.71079	−0.46079	0.09216	3	1.02494	1.53741	1.41271	−0.28254
3	1.04216	1.56324	1.10245	−0.22049	2	0.74240	−0.55546	0.85725	−0.17145
4	0.82167	−0.10245	1.00000		1	0.57095	0.14275	1.00000	

Consequently, for $F = 1$, we find here:

$$A_1A_2 = 0.30000, \quad A_1C_1 = 2.00000, \quad A_2C_2 = 4.00962,$$
$$A_1F = -Y_1/F = -0.57095, \quad A_1H = (1 - Y_1)/F = 0.42905,$$
$$A_2F' = X_4/F = 0.82167, \quad A_2H' = (X_4 - 1)/F = -0.17833.$$

If these results are all divided by 10, we shall obtain the constants of the corresponding optical system for the case when $F = 10$; thus, for $F = 10$ dptr, distances all expressed in centimeters, we have

$$A_1A_2 = 3.00, \quad A_1C_1 = 20.00, \quad A_2C_2 = 40.10, \quad A_1H = 4.29,$$
$$A_1F = -5.71, \quad A_2H' = -1.78, \quad A_2F' = 8.22;$$

these points being ranged along the axis in the order F, A_1, H′, A_2, H, F′, C_1, C_2.

In the next illustration we shall suppose that the *refracting power of the system is negative* ($F = -1$); and in this case the two arbitrary assumptions which we shall make are expressed by the conditions:

$$\phi x = -\frac{1}{30}, \quad \phi = \frac{7 - 6x}{2}.$$

The first condition is equivalent to assuming that the thickness of the lens is $d = 1/20$, (n = 1.5), which is found to be a convenient value to

take for this magnitude in a number of cases which have been investigated. The other condition implies that the absolute term of the quadratic equation in ϵ is missing ($C=0$), and hence that one of the roots is $\epsilon=0$. If $\epsilon=0$, then $R_2=0$, that is, the second face of the lens is a plane surface. Combining the two equations above, we find:

$$x=-0.00944722, \quad \phi=-r_1=+3.52837.$$

Accordingly, the lens in this instance is *planoconcave*; and the actual calculation gives the following results (all distances expressed in centimeters):

$$F=-10\,\text{dptr}:$$

$$A_1A_2=0.50, \quad A_1C_1=-35.28, \quad A_2C_2=\infty,$$

$$A_1H=0.57, \quad A_1F=10.57, \quad A_2H'=-0.43, \quad A_2F'=-10.43;$$

the sequence of points being C_1, F', A_1, H', A_2, H, F.

In the following illustration the first assumption is that the curvatures of the two surfaces of the lens are equal but of opposite signs ($R_1+R_2=0$); in which case, therefore, $\epsilon=-1$ or $C=B-A$. This leads to the following equation between x and ϕ:

$$9x^3-48x^2+75x-28+4\phi=0.$$

If here also (as in the preceding example) we put $c=1/30$ and suppose that the refracting power of the optical system is positive ($F=+1$), then the second condition will be $\phi x=1/30$; and hence, on eliminating ϕ from these two equations, we derive finally an equation containing x only, viz.:

$$135x^4-720x^3+1125x^2-420x+2=0.$$

The only root that is of practical value here is $x=0.004824$, for which

$$\phi=r_1=-r_2=6.9099065.$$

Accordingly, the lens in this case is *equi-biconvex*; and being symmetric on both sides, the X-calculation alone will be sufficient. The result is

$$X_4=-Y_1=0.94970, \quad F=1.$$

The positions of the cardinal points for $F=10$ dptr are given in No. 1 in Table A.

TABLE A. *Exhibiting results of numerical calculations in various special cases, most of which are referred to in the text. (Distances all given in centimeters.)*

$n=1.5$, $F=10$ dptr, $A_1A_2=0.5$ cm

No.	Form of lens	$r_1=A_1C_1$	$r_2=A_2C_2$	A_2H	A_1F	A_2H'	A_2F'	Sequence of points
1	Equi-biconvex	69.099	−69.099	0.503	− 9.497	− 0.503	9.497	C_2, F, H′, A_1, A_2, H, F′, C_1
2	Convex men.	17.98	35.96	0.71	− 9.29	− 0.29	9.71	F, A_1, H′, A_2, H, F′, C_1, C_2
3	Convex men.	24.37	81.91	0.64	− 9.36	− 0.37	9.63	F, A_1, H′, A_2, H, F′, C_1, C_2
4	Double convex	0.5000	− 0.8046	25.500	15.500	−16.667	−6.667	H′, F′, C_2, A_1, $A_2\equiv C_2$, F, H
5	Double convex	0.1667	− 0.1324	9.2667	− 0.7333	30.0000	40.0000	F, A_2, C_1, C_2, A_2, H, H′, F′
6	Concentric	2.5	2.0	2.5	− 7.5	2.0	12.0	F, A_1, A_2, $C_1\equiv C_2\equiv$ H $\equiv$ H′, F′
7	Planoconvex	34.71	∞	0.58	− 9.42	− 3.43	6.57	F, H′, A_1, A_2, H, F′, C_1
8	Convex men.	20.00	45.66	0.65	− 9.35	− 0.32	9.68	F, A_1, H′, A_2, H, F′, C_1, C_2
9	Double convex	80.00	−60.82	0.49	− 9.51	− 0.51	9.49	C_2, F, H′, A_1, H, A_2, F′, C_1

In another illustration the lens is supposed to be a *convex meniscus* for which $R_1>0$ and $R_1=2R_2$ or $\epsilon=0.5$. In this case $A+2B+4C=0$, and, accordingly,

$$9x^3+24x^2-42x-28+16\phi=0.$$

Again assuming that $F=1$ and $\phi x=1/30$, and eliminating ϕ as before, we find in this case:

$$135x^4+360x^3-630x^2-420x+8=0,$$

the required root here being $x=0.01853764$, and hence $\phi=r_1=1.798143$. Here we find:

$$X_4=0.971177, \qquad Y_1=0.928672.$$

The results for $F=10$ dptr are given in Table A, No. 2.

In general, the quadratic equation in ϵ may be written as follows:

$$9\epsilon^2x^3+24\epsilon(1-\epsilon)x^2+\{12(1+\epsilon^2)-51\epsilon\}x-14(1-\epsilon)+4\phi=0.$$

As a final illustration of the method, let us suppose that $\epsilon=0.3$ and $\phi x=1/30$; in which case

$$12.15x^4+75.6x^3-33.3x^2-147x+2=0.$$

The roots of this equation are all real, two of them being positive and the other two negative, viz.: 0.013565, 1.4420, −1.32456 and −6.35323. Only the first, $x=0.013565$, need be considered; the corresponding

value of ϕ is $\phi = 2.457304$. On the assumption that the refracting power of the optical system is positive, the lens is a *convex meniscus* (see Table A, No. 3).

The simplest and most direct procedure (as can be seen from these illustrations) is to assign some value to ϵ in advance; whereas the other method of finding the two values of ϵ corresponding to more or less arbitrary values of ϕ and x is not likely to be so satisfactory from the practical point of view. However, let us proceed now to investigate certain special solutions that throw light on the conditions involved in the problem. Although these cases are doubtless mainly of theoretical interest, incidentally they may lead to results that are not without practical importance.

One of the most obvious methods is to see what will be the effect when one of the coefficients A, B, and C vanishes.

1. If $C = 0$, that is, if

$$2\phi = 7 - 6x,$$

then either $\epsilon = 0$ (and the second surface is plane, as in one of the illustrations given above) or $\epsilon = -B/A$. For example, putting $\phi x = 1/30$ and $F = 1$, we find for $F = 10$ dptr (distances being expressed in centimeters) four cases, as follows:

$C = 0$

$n = 1.5$, $F = 10$ dptr, $d = 0.5$ cm

Form of lens		A_1C_1	A_2C_2	A_1H	A_1F	A_2H'	A_2F'	Sequence
Planoconvex	1	34.71	∞	0.576	− 9.424	− 0.429	9.571	F, A_1, H', A_2, H, F', C_1
	2	0.288	∞	69.423	59.423	−11.904	− 1.904	H', F', A_1, C_1, A_2, F, H
Double convex	1	34.712	−0.290	11.944	1.946	−67.992	−57.992	H', F', A_1, C_1, A_2, F, H, C_2
	2	0.288	−0.096	9.005	− 0.995	52.492	62.492	F, A_1, C_1, C_2, A_2, H, H', F'

Only the first solution ($r_1 = 34.71$ cm, $r_2 = \infty$) would seem to be of any practical value.

2. When $B = 0$, the two roots of the quadratic equation are equal but of opposite signs ($\epsilon = \pm(-C/A)^{1/2}$). Since $24x^2 - 51x + 14 = 0$, either $x = 1.80113$ or $x = 0.32387$. If the refracting power of the optical system is positive, the lens will be double convex or concave meniscus, unless the thickness of the lens is excessive. The meniscus lens can be obtained from the double convex lens or *vice versa* simply by reversing

the second surface or turning it inside out, so to speak. The following four solutions were found for $F=10$ dptr (distances all in centimeters):

$B=0$

$n=1.5$, $F=10$ dptr, $d=0.5$ cm

Form of lens	A_1C_1	A_2C_2	A_1H	A_1F	A_2H'	A_2F'	Sequence of points
Double convex	1.029	−0.428	14.842	4.842	−31.582	−21.582	H′, F′, A_1, C_1, A_2, C_2, F, H
Concave meniscus	1.029	0.428	5.276	− 4.724	8.727	18.727	F, A_1, A_2, C_2, C_1, H, H′, F′
Double convex	0.185	−0.128	9.297	−0.703	42.783	52.783	F, A_1, C_1, C_2, A_2, H, H′, F′
Concave meniscus	0.185	0.128	903.089	893.089	− 9.759	0.241	H′, A_1, C_1, A_2, C_2, F′, F, H

3. Lastly, if $A=0$, then $\epsilon=-C/B$ (for the other case, when $\epsilon=\infty$, $r_2=0$, does not need to be considered). There are three possibilities here, $x=0$, $x=2/3$ and $x=2$.

(*a*) Case when $A=x=0$: then $B=14$, $C=2(2\phi-7)$, $\epsilon=(7-2\phi)/7$, and $\phi x=0$ (unless $\phi=\infty$ which would mean $\epsilon=\infty$ also). Hence, when $x=0$, then $c=0$ for all finite values of ϕ, and in this case the system consists of an *infinitely thin lens* for which

$$X_4=Y_1=1, \quad (c=0);$$

which simply means that the two principal points (H, H′) are a pair of coincident points. The optical system in this case is equivalent to an ordinary system of a single infinitely thin lens of refracting power

$$F=F_1+F_2+F_3+F_4=7(R_1-R_2)/2.$$

(*b*) Case when $A=0$, $x=2/3$: then $B=-28/3$, $C=2(2\phi-3)$, and $\epsilon=3(2\phi-3)/14$. On the assumption that the refracting power of the system is positive, the center of curvature of the first face of the lens coincides with the vertex of the second face (only in case $n=1.5$). An example of a *double convex lens* of this kind is given in Table A, No. 4.

(*c*) Finally, when $A=0$, $x=2$, then $B=8$, $C=2(2\phi+5)$ and $\epsilon=-(2\phi+5)/4$. Accordingly, for $F>0$ and $n=1.5$, we shall have here $d=3r_1$, as in the *double convex lens* in Table A, No. 5.

In conclusion, it may be assumed that ϕ is a function of x such that the expression under the radical, namely, $\Delta=B^2-4AC$, is a perfect square, in which case ϵ will be a rational function of x. In order to study this case, let us put

$$\Delta=(px^2+qx+m)^2+\mu x^4+\alpha x^3+\beta x^2+\gamma x+\delta-48x(x-2)(3x-2)\phi,$$

where, obviously, the eight coefficients denoted by p, q, m, μ, α, β, γ, and δ must satisfy five equations as follows:

$$\mu=144-p^2, \quad \alpha=-2(pq+396)$$
$$\beta=1353-q^2-2pm,$$
$$\gamma=-2(qm+378), \quad \delta=196-m^2.$$

If

$$\phi=\frac{\mu x^4+\alpha x^3+\beta x^2+\gamma x+\delta}{48x(x-2)(3x-2)},$$

then

$$\Delta=(px^2+qx+m)^2$$

will be a perfect square. Three of these eight coefficients may have any arbitrary values whatever (with certain obvious exceptions that would be inconsistent with the above expression for Δ, as, for example, $p=\alpha=0$), and then the other five coefficients will be determined by the relations above specified. Thus, for example, by assuming that two of the coefficients vanish (except only the pairs

$$p=\alpha=0, \quad q=\alpha=0, \quad q=\gamma=0 \text{ and } m=\delta=0,$$

for the reason just mentioned), we may obtain expressions of much simplicity in some instances, although they will not necessarily be of any particular value merely on this account. By way of illustration, suppose that $\mu=\delta=0$; then

$$\phi=\frac{\alpha x^2+\beta x+\gamma}{48(x-2)(3x-2)},$$

$$p=12, \quad \alpha=-24(q+33) \text{ ; and}$$

$$\text{either} \begin{cases} m=+14, \\ \beta=1017-q^2, \\ \gamma=-28(27+q), \end{cases} \quad \text{or} \begin{cases} m=-14, \\ \beta=1689-q^2, \\ \gamma=-28(27-q) \text{ ;} \end{cases}$$

$$\Delta=(12x^2+qx+14)^2.$$

All that remains to be done now is to assign some convenient value to q, as shown in several cases in the subjoined table.

$\mu=\delta=0; \quad p=12$

$m=14$		$m=-14$	
q	ϕ	q	ϕ
-27	$-\frac{3x}{3x-2}$	-21	$-\frac{2(3x-7)}{3x-2}$
-33	$-\frac{3x-7}{2(x-2)(3x-2)}$	$+27$	$-\frac{10x}{x-2}$
-35	$\frac{3x-7}{3(3x-2)}$	-33	$\frac{5(5x-14)}{2(x-2)(3x-2)}$
-43	$\frac{5x-14}{3(x-2)}$	-43	$-\frac{5(6x^2-4x-49)}{6(x-2)(3x-2)}$

Similarly, for the case when $\mu=\alpha=0$, we have the following system of relations:

$$\phi=\frac{\beta x^2+\gamma x+\delta}{48x(x-2)(3x-2)},$$

$$p=\pm 12, \qquad q=\mp 33, \qquad \beta=24(11\mp m)$$

$$\gamma=6(\pm 11m-126), \qquad \delta=196-m^2,$$

the value of m still remaining to be chosen. Or, again, if $\gamma=\delta=0$, then

$$\phi=\frac{x(\mu x^2+\alpha x+\beta)}{48(x-2)(3x-2)},$$

$$q=\pm 27, \qquad m=\mp 14, \qquad \alpha=-18(44\pm 3p),$$

$$\beta=4(156\pm 7p), \qquad \mu=144-p^2,$$

where p now is the one remaining arbitrary coefficient. An endless variety of expressions for ϕ can be obtained in this fashion, all satisfying the condition that $\Delta=(px^2+qx+m)^2$. Of course, other expressions satisfying this same requirement may be obtained without necessarily making one or more of the eight coefficients vanish. Thus, for example, the subjoined table shows the values of p and q on the assumption that $m=14$ and ϕ is a simple linear function of x such that

$$144\phi=ux+v,$$

where u and v must be positive or negative integers:

p	q	u	v	ϕ
24	−51	−432	504	$\frac{7-6x}{2}$
18	−47	−180	420	$\frac{5(7-3x)}{12}$
3	− 9	135	−378	$\frac{3(5x-14)}{16}$
− 3	− 5	135	−462	$\frac{45x-154}{48}$

Some of these expressions for ϕ are worth investigating, although most of them are without any special interest. Consider, for example, the case when

$$\phi=-\frac{3x}{3x-2},$$

for which one of the values of ϵ is

$$\epsilon=-\frac{2}{3x-2}.$$

On the assumption that the refracting power of the optical system in this case is positive, we find that $R_2=F+R_1$ and that $A_1H=A_1H'=A_1C_1=A_1C_2$; accordingly, the lens is a *concentric meniscus* for which the two principal points of the optical system coincide with each other at the common center of curvature of the two faces of the lens (see Table A, No. 6, and also the first numerical example given in this paper).

In case

$$\phi=-\frac{10x}{x-2}, \qquad \epsilon=-\frac{2}{x-2},$$

we find that for $F=1$, that

$$c=\frac{2}{R_1(1+10R_1)}, \qquad R_2=\frac{1+10R_1}{10},$$

$$Y_1=\frac{50R_1^2-15R_1-12}{50R_1^2}, \qquad X_4=\frac{10(10R_1^2+3R_1-2)}{(10R^2{}_1+1)}.$$

For example, when $F=10$ dptr and $d=1$ cm, we find here:

$$A_1C_1=5.95, \quad A_2C_2=5.61, \quad A_1H=2.63,$$

$$A_1F=-7.37, \quad A_2H'=0.93, \quad A_2F'=10.93$$

(distances all given in centimeters).

Again, if

$$\phi = -\frac{2(3x-7)}{3x-2},$$

one of the solutions for $F=1$ is contained in the following set of formulas:

$$c = \frac{2(7R_1+1)}{3R_1(2R_1+1)}, \qquad R_2 = -\frac{3(2R_1+1)}{5},$$

$$Y_1 = \frac{6R_1+1}{R_1}, \qquad X_4 = -\frac{2R_1^2-143R_1-22}{3(2R_1+1)^2};$$

and here for $F=10$ dptr and $d=1$ cm the numerical results (in centimeters) are as follows:

$$A_1C_1 = -69.29, \quad A_2C_2 = -23.43, \quad A_1H = 0.71,$$
$$A_1F = -9.29, \quad A_2H' = 270.60, \quad A_2F' = 280.60;$$

the sequence of points being C_1, C_2, F, A_1, H, A_2, H′, F′.

Many other examples might be given, but the above are perhaps sufficient for the purposes of this paper. The actual optical system here considered is one which seems to have very little utility, as was admitted at the outset; but the problem itself illustrates methods of geometrical optics which often have to be employed in practical work and which sometimes puzzle even the trained investigator. For that reason, if for no other, a student may find it exceedingly useful to see how these methods are applied in this comparatively simple case.

SUPPLEMENTARY NOTE CONCERNING A SO-CALLED "THICK MIRROR" CONSISTING OF A SINGLE LENS.—A "thick mirror" is a term used by Dr. Searle* to describe any symmetrical optical system in which the rays, after having traversed the system and arrived at the last surface, are there reflected back through the entire anterior part of the system in the opposite direction, emerging finally into the medium where the luminous object is situated. For example, the so-called *Purkinje reflex images* in the human eye, which are of such fundamental importance in ophthalmometry, are formed by optical systems that are really "thick mirrors" in the sense in which that term is defined above. The anterior part of the system comprises the entire system except the mirror formed by the last surface where the rays are reflected.

* G. F. C. Searle: The determination of the focal length of a thick mirror. Proc. Cambr. Phil. Soc., *18*, Part 3, pp. 115–126; 1915. See also Southall's *Mirrors, Prisms, and Lenses*, Revised edition (The Macmillan Company, New York), pp. 376–384; 1923.

Using a method and notation precisely analogous to that employed in the main portion of this paper, let $F_1 = F_3$ denote the refracting power of the anterior system; moreover, let $F_2 = -2nR_2$ denote the reflecting power of the surface at which the rays are reflected, where n denotes the index of refraction of the medium in contact with this surface and R_2 denotes the curvature of the surface; and, finally, let

$$c = \frac{H'_1A_2}{n}$$

denote the reduced interval from the second principal point of the anterior system (H'_1) to the vertex A_2 of the reflecting surface. In terms of the magnitudes thus defined, the reflecting power of the "thick" mirror (F) and the positions of its cardinal points can be calculated by means of the following system of formulas:

$$\begin{aligned} X_1 &= 1, & F_{11} &= F_1 = F_3, \\ X_2 &= X_1 - c \cdot F_{11}, & F_{12} &= F_{11} + X_2 \cdot F_2, \\ X_3 &= X_2 - c \cdot F_{12}, & F_{13} &= F_{12} + X_3 \cdot F_3 ; \end{aligned}$$

$$F = F_{13}, \quad H_1F = H_1F' = -X_3/F,$$

$$H_1H = H_1H' = (1 - X_3)/F,$$

$$H_1K = -(1 + X_3)/F.$$

The capital Roman letters above designate the positions of the principal points (H, H′) and the focal points (F, F′) of the "thick" mirror, the first principal point (H_1) of the anterior part of the system, and the center (K) of the equivalent "thin" mirror; for it appears from these formulas that there is a single reflecting surface or so-called "thin" mirror which is optically equivalent to the "thick" mirror, inasmuch as the pair of coincident points (H, H′) may be considered as the vertex, and the other pair of coincident points (F, F′) as the focus, of an equivalent "thin" mirror of radius $HK = -2/F$.

The above formulas are perfectly general. Now let us assume that the entire optical system (as in the case considered in the main part of this article) is composed of a *single lens*, made of glass of index n and surrounded by air; so that the anterior part of the system is a single spherical refracting surface of curvature R_1, namely, the first surface of the lens, while the second surface of curvature R_2 is the reflecting surface. In this simple special case, therefore, the points designated by H_1, H'_1 are a pair of coincident points both situated at the vertex (A_1)

of the first surface of the lens; and if $d = A_1A_2$ denotes the thickness of the lens, then $c = d/n$ denotes its reduced thickness. Accordingly, applying the preceding formulas to the case of a "thick" mirror consisting of a single lens, we must have here:

$$F_1 = F_3 = (n-1)R_1, \quad F_2 = -2nR_2,$$
$$A_1F = A_1F' = -X_3/F, \quad A_1H = (1-X_3)/F, \quad A_1K = -(1+X_3)/F.$$

If (as in the case of the lens considered in the main part of this article) we assume here also that $n = 1.5$ (as will be assumed throughout the discussion from this point), then

$$F_1 = F_3 = R_1/2, \quad F_2 = -3R_2, \qquad (n = 1.5).$$

Introducing, as before, the symbols

$$\epsilon = \frac{R_2}{R_1}, \quad \phi = \frac{F}{R_1}, \quad x = cR_1,$$

we find in this case:

$$\phi x = cF, \quad \epsilon = \frac{2(2-x-2\phi)}{3(x-2)^2}.$$

If the equivalent "thin" mirror is *concave* ($F > 0$), ϕ and x (or R_1 and x) must have like signs; whereas for a *convex* mirror ($F < 0$), these two magnitudes must have opposite signs. In particular, if $F = +1$ (concave mirror), then $\phi = r_1 = 1/R_1$; and if $F = -1$ (convex mirror), then $\phi = -r_1$.

The *form of the lens* will be determined by the signs of R_1 and x, as exhibited in the subjoined table:

Form of Lens

	$R_1 > 0$	$R_1 < 0$
$\epsilon < 0$	Double convex	Double concave
$\epsilon = 0$	Planoconvex ($R_2 = 0$)	Planoconcave ($R_2 = 0$)
$\epsilon > 0$	Meniscus: (*a*) Convex meniscus ($\epsilon < 1$) (*b*) Concave meniscus ($\epsilon > 1$)	Meniscus: (*a*) Convex meniscus ($\epsilon > 1$) (*b*) Concave meniscus ($\epsilon < 1$)

Putting $\phi x = 1/k$, where k is an arbitrary number that may have any assigned value, positive or negative, depending only on the other prescribed conditions and practical considerations in any particular in-

stance, we may eliminate ϕ and obtain the following cubic equation with respect to x:

$$3\epsilon x^3-2(6\epsilon-1)x^2+4(3\epsilon-1)x+4/k=0.$$

While we have not space here to pursue this study further, attention may be directed briefly to several special cases, by way of illustration:

1. When $x=0$, then $c=0$ (*infinitely thin lens*), and hence $\phi=1-3\epsilon$. If, therefore, $F=1$, then $X_3=1$, $A_1F=-1$, $A_1H=0$, $A_1K=-2$.

2. When $\epsilon=0$, the reflecting surface is plane ($R_2=0$); and (supposing that the lens is not infinitely thick) $x=2(1-\phi)$. For example, if $F=1$, then $c=-2(1-R_1)/R_1^2$, and $A_1F=(R_1-2)/R_1$.

3. When $\epsilon=-1$, the lens is symmetric ($R_1+R_2=0$). For example, for a symmetric double convex lens of thickness equal to 1 cm and with radii each equal to 39.41 cm, the reflecting power will be that of a concave mirror for which $F=10$ dptr, $A_1H=0.67$ cm, $A_1K=-19.33$ cm.

4. When $\epsilon=2$, we find $6x^3-22x^2+20x+4\phi x=0$; and if we assume that $\phi x=1/15$, one of the roots of the cubic equation will be found to be $x=-0.013\ 142\ 65$. In this case the lens is a *convex meniscus* and if $F=10$ dptr, then

$$A_1A_2=1\text{ cm},$$
$$A_1C_1=-50.725\text{ cm},\quad A_2C_2=-25.363\text{ cm},$$
$$A_1H=0.662\text{ cm},\quad A_1K=-19.338\text{ cm}.$$

DEPARTMENT OF PHYSICS,
COLUMBIA UNIVERSITY,
NEW YORK, N. Y.
SEPT. 1, 1927.

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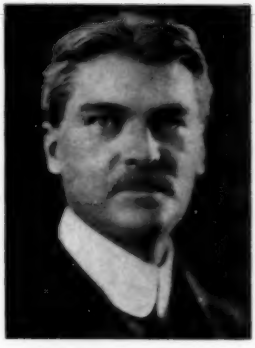

ADOLPH LOMB
Rochester, N. Y.
TREASURER

OFFICERS OF THE OPTICAL SOCIETY OF AMERICA
1927–28

TWELFTH ANNUAL MEETING OF THE OPTICAL SOCIETY OF AMERICA, SCHENECTADY, N. Y., OCTOBER 20, 21, 22, 1927

The Twelfth Annual Meeting of the Optical Society was held at Union College in Schenectady, N. Y., October 20, 21, 22, 1927. About 75 members registered in attendance. The meeting included an address of welcome by President C. E. Richmond of Union College; the reading of technical and scientific papers by members of the Society; a dealers' exhibit of optical apparatus; a complimentary dinner by the General Electric Company and a tour of inspection over the Research Laboratory and Plant of this Company; the annual business meeting of the Society; two invited addresses; the address of the retiring President and a banquet at the Van Curler Hotel.

The complimentary dinner at the Plant of the General Electric Company was a thoroughly enjoyable feature, while the exhibits and demonstrations in the Research Laboratory and the Industrial Plant, to which Thursday afternoon was devoted, were in many cases very striking and impressive.

The optical exhibit was on display throughout the session at the Physics Building of Union College and attracted great interest. Apparatus and optical goods were exhibited by the following firms:

R. C. Burt, Pasadena, Calif.—Stabilized oscilloscope, photoelectric cells.

Bausch & Lomb Optical Co., Rochester, N. Y.—New Universal spectrophotometer and other optical instruments.

Corning Glass Works, Corning, N. Y.—Some special optical glass.

E. Leitz, Inc., New York—New bifocal microscopes and other instruments.

The Thermal Syndicate, Inc., Brooklyn, N. Y.—Vitrosil ware.

Following an address of welcome by President C. E. Richmond of Union College, an invited paper was delivered on Thursday evening by Dr. K. S. Lashley on "Cerebral Function in Vision."

The Business Meeting on Friday was preceded by an invited address by Dr. C. G. Abbot on "Optics, The Key of Astronomy."

The Presidential Address of the retiring President, Dr. W. E. Forsythe, on "Temperature Radiation" was delivered after the Business Meeting.

The Banquet at the Van Curler was enlivened by short talks by Dr. W. R. Whitney, Dr. C. G. Abbot, Professor Ernest Merritt and Professor Frederick Bedell. The talks were followed by a demonstration by Dr. John B. Taylor of the transmission of music over a beam of light.

The minutes of the Business Meeting of October 21, and the abstracts of papers follow:

Minutes of the Meeting of the Optical Society, October 21, 1927

Report of the Secretary. There were elected during the year 22 new associate members as follows:

E. Hetle, M. S.—Teacher of Physics, Northfield, Minn.

T. R. Wilkins, Ph. D.—Prof. of Physics, Univ. of Rochester.

C. del Rosario, M. S.—Sterling Fellow—Yale University (1927).

Henry Miller, Ph. D.—Physicist—Res. Lab. G. E. Co., Cleveland, Ohio.

W. W. Merrymon, Ph. D.—Physicist—Res. Lab. Westinghouse Lamp Co., Bloomfield, N. J.

P. K. Taylor, M. S.—Instructor in Physics, Northwestern U., Evanston, O.

C. T. Knipp, Ph. D.—Prof. of Physics, Univ. of Illinois.

R. E. Lofton, A. B.—Assoc. Physicist, Bureau of Standards.

F. A. Molby, Ph. D.—Prof. of Physics, W. Va. Univ.
G. M. Shrum, Ph. D.—Asst. Prof. of Physics, Univ. of Brit. Col.
F. M. Kannenstine, Ph. D.—Director of Res. Lab. Geo-Physical Corp., Bloomfield, N. J.
F. W. Loomis, Ph. D.—Assoc. Prof. of Physics, N. Y. Univ.
H. A. Wilson, Ph. D.—Prof. of Physics, Rice Institute, Houstin, Tex.
R. J. Lang, Ph. D.—Assoc. Prof. of Physics, Univ. of Alberta, Edmonton, Alta.
H. F. Raess, Photographic Lab. Asst., Warner Research Lab., N. Y. C.
C. L. Treleaven, M. A.—Instructor in Phys., Columbia, Univ.
I. Roman, Ph. D.—Physicist—Geophysical Corp., St. Louis, Mo.
Miss Dorothy Nickerson—Color Technologist, Bur. of Agr. Economics, Washington, D. C.
L. S. Barnes, A. B.—Instructor of Math., Lehigh University.
R. A. Sawyer, Ph. D.—Assoc. Prof. of Phys., Univ. of Michigan.
W. F. Boiler, M. D., M. S.—Physician, Iowa City, Iowa.
R. E. Martin, Ph. D.—Asst. Prof. of Physics, Lehigh University.

The Council in its meeting of October 20th elected to regular membership the following:

C. T. Knipp	H. A. Wilson
F. A. Molby	R. J. Lang
C. M. Shrum	Irwin Roman
F. W. Loomis	R. A. Sawyer

There have been 11 resignations during the year and one death that of C. P. Lewis of the University of California. The membership list now stands at 453 as against 443 of October 1926; 390, October 1925 and 360, October 1924.

The following actions of the Executive Council were announced by the Secretary:

(1) An amendment of the By-Laws: By-Law IV, Section 1, instead of reading "Members-at-Large of the Council shall be elected for a two-year term and shall be eligible for but one re-election" to read "Members-at-large of the Council shall be elected for a two-year term. After a service of two consecutive terms members-at-large of the Council shall be eligible for re-election only after an interval of two years."

(2) Election of Associate editors of the Journal for a six-year term beginning January 1, 1928 as follows:

S. Dushman, K. S. Gibson, L. O. Grondahl,
P. E. Klopsteg, W. B. Rayton, H. B. Williams

(3) Re-election of Dr. F. K. Richtmyer as Managing Editor of the Journal for a term of six years beginning January 1, 1928.

(4) Participation of the Optical Society, by invitation, in the meeting of the American Society of Mechanical Engineers on December 7th in New York, the participation to take the form of three addresses, viz.,

W. P. Davey, "X-Ray Investigations of Structural Materials."
C. E. K. Mees, "Applications of Photography in Engineering."
F. F. Lucas, "Photomicrography."

(5) Approval of continuance of the practice of holding a joint meeting in February in New York with the American Physical Society.

(6) Acceptance of an invitation of the Director of the Bureau of Standards, Dr. Burgess, to hold the next annual meeting of the Optical Society in Washington next October, the exact date to be determined and announced later.

A report of the financial condition of the Society was made by the Treasurer, Mr. Adolph Lomb.

The Secretary announced the result of the election of officers by mail ballot as follows:
President—I. G. Priest.
Vice President—L. A. Jones.

Members of the Executive Council—L. R. Ingersoll, A. H. Pfund, W. F. Meggers, P. E. Klopsteg.

All of the above were elected for a two year term beginning January 1, 1928.

The Secretary announced for purpose of record in the minutes that a very successful joint meeting of the Optical Society with the American Physical Society was held on Feb. 25, 26, 1927 at Columbia Univesity in New York. The program of this meeting was published in J.O.S.A. and R.S.I., 14, June 6, 1927.

The following motion offered by Dr. Frederick Bedell was unanimously adopted:

"Resolved that the Optical Society express its appreciation of the hospitality extended to it by the Department of Physics of Union College and by the Research Laboratory of the General Electric Company."

CHARLES C. EIDWELL, *Secretary*

Paper presented at Sessions of the Society Oct. 20, 21, 22, 1927

1. Deane B. Judd **Bureau of Standards**

SATURATION OF COLORS DETERMINED FROM THE VISUAL RESPONSE FUNCTIONS

It has been discovered empirically from hue discrimination data that the discriminative difference, or differences in "sensation," between many pairs of colors is proportional with good approximation to the difference in their r-coördinates, or to the difference in their g-coördinates, whichever difference is the greater. This paper extends to saturation differences the law which holds for hue differences, and intends to show how successful is this law in representing saturation differences.

If it be agreed that the "saturation" (S) of the color (r, g, b) is to be correctly evaluated by the "sensation" difference between that color and "white" (1/3, 1/3, 1/3), we may hopefully (from the hue discrimination law) write:

$$S=(c-1/3)/\mathrm{K}$$

where: $c-1/3$ = the absolute value of $r-1/3$ or $g-1/3$, whichever absolute value is greater, and K = a constant, approximately equal to 1/100 if S is expressed in "least perceptible differences."

The purity of the color (r, g, b) may readily be calculated from the values of r, g, and b[1]; hence, there exists a way to obtain for any color directly from the visual response functions both that color's purity and a number which we could hope might be its saturation. The variation of saturation with colorimetric purity, known as a saturation scale, may thus be computed from the visual response functions provided the law, above stated, represents saturation differences as well as hue differences. The saturation scale has been experimentally determined for many hues and by a number of observers.[2] These two scales will be compared. A comparison will also be made with the results of Gottlieb[3] on sensibility to saturation change.

Bibliography:
1. Judd, J.O.S.A. & R.S.I., *13*, p. 134; 1926.
2. Nutting, J.O.S.A., *4*, p. 62; 1920; Jones and Lowry, J.O.S.A. & R.S.I., *13*, p. 25; 1926; Judd, J.O.S.A. & R.S.I., *14*, p. 470; 1927.
3. Gottlieb, Wien. Ber., IIA, *126*, p. 1299; 1917.

2. Deane B. Judd **Bureau of Standards**

SENSIBILITY TO COLOR CHANGE DETERMINED FROM THE VISUAL RESPONSE FUNCTIONS; EXTENSION TO COMPLETE AND PARTIAL DICHROMASY

It has been demonstrated[1] that, with normal trichromatic vision, the discriminative difference, or difference in "sensation," between most color pairs is proportional with good

approximation to the difference in their r-coördinates or to the difference in their g-coördinates, whichever difference is the greater. This empiric law, formally stated, is:

$$c-c'=\mathrm{K}(E-E'),$$

where: $E-E'$ = the "sensation" difference between colors (r, g, b) and (r', g', b'), $c-c'$ = the absolute value of $r-r'$ or $g-g'$, whichever absolute value is greater and

K = a constant approximately equal to 1/100 if $E-E'$ is expressed in "least perceptible differences."

In dichromatic vision (the two most common forms of which are protanopy, or red-blindness, and deuteranopy, or green-blindness) only two coördinates are needed for the specification of a color,—g and b for red-blindness, r and b for green-blindness, and r and g for blue-blindness or tritanopy. The empiric law, stated above, is formally applicable to dichromasy without change. One purpose of this paper is to discover whether the formal solution by this law empirically derived from data on trichromats corresponds to the known powers of color discriminations exhibited by dichromats.[2] To this end are compared the formal and experimental curves for dichromats of $(d\lambda/dE)_{p=0}$ and $(d\lambda/dE)_{p=1}$ as functions of λ. (λ = dominant wave length, p = colorimetric purity.)

Further, the curve of $(dp/dE)_{p=0}$ against λ for normal trichromats, determined by Priest and Brickwedde,[3] is shown to be much more closely duplicated by the empiric formula if a visual mechanism which is four-fifths blue-blind be assumed instead of the normal trichromatic mechanism. This suggests the conclusion that the normal fovea is nearly, if not totally, blue-blind or tritanopic.

Bibliography:

1. Judd, J.O.S.A. & R.S.I., *14*, p. 470; 1927; also preceding abstract.

2 Brodhun, Zs. f. Psych. u. Physiol. d. Sinnesorgane, *3*, p. 104; 1892; Koenig, u. Dieterici, Zs. f. Psych. u. Physiol. d. Sinnesorgane, *4*, Section 7, 1892; Steindler, Wien. Ber., IIA, *115*, pp. 39–62; 1906.

3. Priest and Brickwedde, J.O.S.A. & R.S.I., *13*, p. 306; 1926.

3. Irwin G. Priest and K. S. Gibson — Bureau of Standards

STANDARDIZING THE RED AND YELLOW LOVIBOND GLASSES

The intent of the Lovibond color scales for each series of glasses (red, yellow, or blue) may be expressed by the additivity condition,

$$(\sum N)_1=(\sum N)_2,$$

where $(\sum N)_1$ and $(\sum N)_2$ are the sums of the Lovibond scale numerals attached respectively to two combinations of glasses such that: (1) they contain equal numbers of glasses (non-selective glass of nearly zero absorption being introduced if needed), and (2) they evoke the same color under like conditions.

Among actual glasses, large deviations from the additivity condition are frequent, as are differences between glasses nominally alike.

The purpose of the present standardization is to establish scales satisfying the additivity condition. The procedure has been:

(1) To accept a given *set* of glasses, obtained directly from the makers, as establishing (implicitly) the unit and the *general* nature of the scale.

(2) To derive the unit from these actual glasses.

(3) To assign the glasses new numerals such that the additivity condition is satisfied.

This standardization is based upon the spectral transmission of the glasses, and has been carried out *via* the trilinear coördinates computed on the basis of the "excitations" given by the O. S. A. Committee on Colorimetry. We have computed the trilinear coördinates (r, g, b), the dominant wave length (λ), the purity (P), and the sunlight transmission (T).

The yellow glasses constitute preeminently a saturation scale. At first thought one might choose purity as the most suitable variable in terms of which to make the calibration. We

have found $1/b$ to be a more convenient parameter. The correct numeral for a yellow glass is found from $1/b$, but the values of Λ and T must also fall within certain tolerances.

Since they are so used in the oil industry, the red glasses have been calibrated in combination with 35-yellow. This combination (35-yellow with variable red) gives preeminently a hue scale. The ratio r/g has proved to be the most suitable parameter for calibration purposes. The correct numeral for a red glass is found from r/g, but the values of P and T must fall within certain tolerances.

Having established by computation numerals for many glasses covering the scales, we have checked their additivity by direct comparison of colors (hue and saturation, at equated brilliance) in a Martens Photometer.

Bibliography:
Gibson and Harris, B. S. Sci. Pap. *547*. (Abstract in J.O.S.A. & R.S.I., *12*, p. 481; 1926.)
Report of Colorimetry Committee, J.O.S.A. & R.S.I., August, 1922.
Priest, J.O.S.A. & R.S.I., *9*, pp. 503–520; November, 1924.
The present paper will appear in full in B S. Scientific Papers.

4. Irwin G. Priest **Bureau of Standards**

PRELIMINARY DATA ON THE LEAST PERCEPTIBLE DIFFERENCE IN DOMINANT WAVE LENGTH BY THE METHOD OF RIGHT AND WRONG ANSWERS

The mere phrase, "least perceptible difference," is only a nominal definition; so soon as one attempts to set such a difference experimetally, to the full significance of the term "least," he feels that he is following a will-o'-the-wisp. In order to state more precisely what was to be determined, I proposed some years ago that the problem be attacked by the "method of answers" and the results stated in terms of the experimental probability of perceiving a given difference.[1] A great many years must elapse before comprehensive definitive data can be compiled by this time-consuming method. In the meantime, fragmentary data such as presented herewith may be of interest as indicating the nature of the result to be expected, and the very small difference which is perceptible. These data are merely the by-product of another investigation (the calibration of Lovibond glasses). The colors viewed are evoked not by homogeneous light, but by light of a spectral distribution giving (by computation O.S.A. excitations)[2] a dominant wave length about 585 mμ and purity about 0.93 (Lovibond 35 yellow, 7.6 red.)

EXPERIMENTAL METHOD AND CONDITIONS

Observer, with eyes light adapted, but protected from disturbing side lights, views a 6° circular field, divided vertically in halves, with dark surrounding field. (Martens Photometer). Lovibond glasses 35 yellow and 7.6 red[3] over exit pupil modify daylight illumination in both halves of the field alike, and determine the color for which discrimination is tested. A fractional Lovibond red glass inserted by the conductor of the experiment introduces an increment ($\Delta\Lambda$) in dominant wave length in *either* the right or left half. The observer knows that one half or the other has been so changed; but he has no means (other than hue discrimination) of knowing *which*. He matches brightness, himself, and is required to answer "right" or "left" in each of 10 or 20 trials. Observers were always urged to report their slightest suspicion even if very doubtful. (Only 2 observers out of 30 have absolutely insisted on reporting "matched" a few times. In computing results, such "matches" are arbitrarily divided equally between "correct" and "incorrect.") Actual number of presentations on right equals actual number on left.

DATA FOR DOMINANT WAVE LENGTH $= 585 \pm 1$ mμ AND PURITY $= 0.93 \pm 0.01$.

$\Delta\Lambda$ is diff. in dom. wave length. C is number of correct answers. I is number of incorrect answers. M is number of times observer reported "matched". $P = C + (C + I + M)$ is experimental probability of correct answer.

Detailed Synopsis for Four Approximately Normal Trichromats.

Observer	Date	ΔΔ	*C*	*I*	*M*	*P* (literal)		*P* (fictional)	
D. B. Judd	5–11–27	0.2 mμ	20	0	0	1.0		1.0	
	5–11–27	0.1	20	0	0	1.0		1.0	
	9– 9–27	0.1	20½	0	0	1.0		1.0	
K. S. Gibson	5–11–27	0.2	14	1	5	0.7	*Considering*	0.82	*By dividing*
	5–11–27	0.1	10	2	8	0.5	*only*	0.70	*"matches"*
	9– 9–27	0.1	14	3	3	0.7	*"correct" answers*	0.78	*equally between*
G. K. Walker	8–26–27	0.2	10	0	0	1.0		1.0	*"correct" and*
	8–26–27	0.1	7	3	0	0.7		0.7	*"incorrect."*
	8–26–27	0.1	9	1	0	0.9		0.9	
J. O. Riley	9– 9–27	0.1	14	6	0	0.7		0.7	

Condensed Snyopsis for 17 other Observers Selected as Normal by Stilling's Chart Test.

ΔΔ		0.2 mμ	0.3 mμ	ΔΔ		0.2 mμ	0.3 mμ
Average	*P*	0.75	0.95				
Minimum	*P*	0.4	0.9	Number of observers obtaining min.	*P*	2	9
Maximum	*P*	1.0	1.0	Number of observers obtaining max.	*P*	2	8

(Three observers who read Stilling's Charts with difficulty obtained slightly lower values of *P*.) The "blind man's score" (*P* obtained by pure chance without aid of hue discrimination) = 0.5. (More detailed presentation of the data on which the above abstract is based will appear in *Oil and Fat Industries* under the title, "Tests of Color Sense . . . . Sensibility to Change in Lovibond Red . . . .")

Bibliography:

1. Priest, Appendix to B. S. Sci. Pap., *417* and also J.O.S.A. & R.S.I., 7, pp. 99–100; 1923.
2. Colorimetry Report, J.O.S.A. & R.S.I., Aug., 1922.
3. Gibson and Harris, B. S. Sci. Pap., *547*, particularly Fig. 9.
4. Priest and Judd, J.O.S.A. & R.S.I., *14*, pp. 137–138; Feb., 1927.
5. Priest, J.O.S.A. & R.S.I., *14*, pp. 138–139; Feb., 1927.

(*Correction to last reference.* Redetermination of spectral transmission of glass B and more reliable direct measurements of the difference show that the difference in dominant wavelength given as 0.2 mμ is actually about 0.44 mμ.)

5. Progress Report on Geometrical Optics.

H. F. Kurtz

6. Henry F. Kurtz

Bausch & Lomb Optical Company

Orthostereoscopic Reproduction Under Given Magnification, of an Object at a Finite Distance

"Orthostereoscopic Reproduction" is here intended to convey the idea that the composed view received by the observer shall have all dimensions in three coördinates equal to the same dimensions measured on the object itself, or that there shall be a constant factor of proportionality for three coördinates between the same dimensions in object and view. "Magnification" is the factor of proportionality between the dimensions of object and view. Correct control of perspective is essential, and therefore the assumption of a "normal viewing distance" is essential. This may be the shortest distance at which comfortable fusion and convergence may be accomplished, and for ordinary small objects fifteen inches has been found satisfactory.

"The normal viewing distance," the interpupillary distance and the magnification desired are the three quantities which determine the conditions of the photographic "set-up" and the design of the stereoscope. The magnification may lie all in the photographic system, all in the stereoscopic system, or may be divided between them. The choice for the distribution of the magnification is determined by practical considerations as for instance grain, size, desirability of unit size photography for purposes of measurement, etc. Equations for the focal length of photographic lenses, lens to object distance, lens to plate distance, lens separation, focal length of stereoscope lenses, optical length of stereoscopes, etc., were given.

7. **Otto Sandvik** **Kodak Research Laboratories**

THE DEPENDENCE OF THE RESOLVING POWER OF A PHOTOGRAPHIC MATERIAL UPON THE RELATIVE CONTRAST IN THE SUBJECT

The resolving power of a photographic material may be defined qualitatively as the ability to show fine detail in the picture. It is defined numerically as the number of lines and spaces per millimeter which it resolves. This definition, however, is rather inadequate since resolving power depends on many factors, such as the ratio of the width of the line to the width of the space, the color temperature of the light image, or the wave length where monochromatic radiation is in question, and the relative contrast in the subject.

The present paper gives some results of an experimental investigation of the dependence of resolving power upon the relative contrast in the test object, where relative contrast is defined as the ratio of the photographic intensities of two adjacent small areas to be resolved.

The method of investigation was to photograph in a reducing camera a series of parallel line test objects differing only in relative contrast, and by microscopic examination of the developed photographic images to determine the maximum resolving power for the respective objects.

The results show that the resolution changes very rapidly with contrast at low contrast values. Thus, with no resolution at unit contrast, the resolving power reaches approximately 65% of its maximum value for a test object density of 0.5, that is, a transmission through the opaque spaces of 31.5% or a relative contrast of 3.17; and 87% of its maximum value when the test object density is 1.0, transmission 10% and relative contrast is 10. The maximum value of resolving power is reached when the test object has a relative contrast of approximately 100 to 200.

The present paper will appear in full in J.O.S.A. & R.S.I.

8. **Arthur C. Hardy** **Mass. Institute of Technology**
Frederick W. Cunningham **Mass. Institute of Technology**

A RECORDING PHOTOELECTRIC SPECTROPHOTOMETER

This instrument has been designed primarily for the rapid measurement of the color of reflecting surfaces. The specimen to be measured and a piece of magnesium carbonate are illuminated from opposite sides of a ribbon filament tungsten lamp. Beams of light from the specimen and from the magnesium carbonate are alternately admitted to the slit of the dispersing system. Before reaching the slit, both beams traverse symmetrical paths except that one beam is reflected at a second-surface silvered mirror. After passing the slit, both beams traverse the same path. A photoelectric cell receives light from a small fraction of the spectrum produced by the dispersing system.

When the two beams are of unequal intensity, an alternating current is generated in the photo-electric cell. This current is amplified by pliotrons and rectified by a synchronous commutator. The rectified current operates a galvanometer relay which controls a reversing motor. This motor controls a diaphragm which varies the illumination on the magnesium carbonate, and automatically equalizes the intensities of the two beams.

A pencil operated from the mechanism controlling the diaphragm draws a trace on a rotating drum. The rotation of this drum is used to change the wave length of the light admitted to the photoelectric cell. Thus the instrument automatically plots the spectrophotometric curve rapidly, and with a precision exceeding that of visual instruments.

9. **Loyd A. Jones** **Kodak Research Laboratories**

LIGHT FILTERS FOR THE ISOLATION OF NARROW SPECTRAL REGIONS

Data were given relative to a series of light filters by means of which relative narrow bands of radiation may be isolated. These bands are spaced through the region from 300 to

900 mμ. Dyed gelatin, colored glass, and solutions of stable inorganic salts are used in the construction of these filters.

The present paper will appear in full in J.O.S.A. & R.S.I.

10. J. W. Forrest **Bausch & Lomb Optical Company**

A UNIVERSAL SPECTROPHOTOMETER

The report described a new spectrophotometric arrangement designed to be universal in its application. A sphere was used to provide for the illumination of opaque bodies. When measurements on transparent bodies are to be made, the sphere may be replaced by a lamp house giving direct illumination.

With the need of the dye industry in mind an attachment was designed which provides for liquid columns easily and quickly adjustable in length.

11. Progress Report on Physical Optics. **W. P. Davey**

12. Ludwick Silberstein **Research Laboratory, Eastman Kodak Company**

REFRACTION OF SKEW RAYS BY PRISM, TREATED VECTORIALLY

The incident ray, of any orientation, being represented by the unit vector $\mathbf{r}$, the refracted and ultimately emergent ray is expressed by the vector formula $\mathbf{r}' = \mathbf{r} + g_1\mathbf{n}_1 + g_2\mathbf{n}_2$ (given some time ago in "Simplified Method"), where $\mathbf{n}_1$, $\mathbf{n}_2$ are the normals of the two faces of the prism and g_1, g_2 are scalar factors; in obvious notation, $g_1 = \cos i_1 - \mu \cos i_1'$, etc. From this simple formula the two angular coördinates of the emergent in terms of those of the incident ray are now most directly deduced. If θ be the inclination of the incident ray to the edge of the prism and ϕ the angle between the first face of the prism and the plane through the edge parallel to the incident ray, then, for the emergent ray, $\theta' = \theta$ and, the scalar g_2 being readily eliminated,

$$\cos(\phi' - \epsilon) = \cos(\phi - \epsilon) - \frac{g_1 \sin \epsilon}{\sin \theta},$$

where ϵ is the refracting angle of the prism. The factor g_1 is written explicitly in terms of the incidence data. The advantages of the vectorial as compared with the usual treatment are brought out. The limiting condition, beyond which there is total reflection, is derived and discussed.

Bibliography:

L. Silberstein, Simplified Method of Tracing Rays, etc., Longmans, London 1918.

The present paper will appear in full in J.O.S.A. & R.S.I.

3. Wm. G. Exton, M.D. Laboratory of the Prudential Insurance Company of America

IMPROVED EXTINCTION METHOD OF MEASURING TURBIDITY

Several instruments designed to measure turbidity by a visual acuity (vanishing point) criterion have been previously shown the Society. These function best with a glass target having a translucent peripheral field enclosing a circular platinized middle portion with a transparent slit in its center. Turbidity is measured by the depth of sample at which the target slit vanishes.

In the later instruments besides the light incident to the target slit and the diffuse incident light the turbid sample is also illuminated by Tyndall beam. By integrating light in this way a unique compensatory effect is achieved which makes the method a nearly pure measure of the obstructance offered by the particles in suspension to the rays of light forming the target

image. The method is practical for samples of limited quantity and compares favorably with existing turbidity in range and sensitivity.

In addition to avoiding all troubles connected with comparison standards by reading the sample directly, experience shows the method approximates simple proportionality closer and is much less affected by differences in light, color, flocculation and particle size than existing methods. In routine work the method has given satifactory reproducibility in many different hands and is remarkably free from disturbances of calibration.

Bibliography:
Exton, W. G. Proc. Soc. Amer. Bacter., Dec.; 1921.
Exton, W. G. Proc. Opt. Soc., Oct. 24, 1922; J.O.S.A., *6*, p. 414, 1922.
Exton, W. G., J.O.S.A., *11*, p. 126; 1925.

The present paper will appear in full in J.O.S.A. & R.S.I.

14. J. B. Austin and F. C. Brickwedde **Low Temperature Laboratory, Bureau of Standards, Washington, D.C.**

THE ULTRAVIOLET ABSORPTION SPECTRA OF BENZENE, TOLUENE, AND ORTHO, META AND PARA XYLENES AT LOW TEMPERATURES

The ultraviolet absorption spectra to 2300A have been obtained of (a) liquid benzene, toluene, an ortho, meta and para xylenes at room temperatures and liquid air temperatures and (b) solid benzene, toluene, and ortho, meta and para xylenes and mesitylene at liquid hydrogen temperatures.

The spectra were photographed with a Hilger E2 quartz spectrograph using an underwater spark as the source of light. The cell was made by placing two plane quartz plates in contact and immersing one end in a sample of the compound to be studied, the thickness of the absorbing layer being that which flowed up between the plates held together by capillary attraction. Thicker films produced complete absorption in the region of banded absorption. Absorption was studied at room, liquid air and liquid hydrogen temperatures. For low temperature work the cell was placed in a transparent quartz dewar flask partially filled with liquid air or liquid hydrogen and the dewar was mounted in front of the slit of the spectrograph. At liquid air temperatures all the compounds studied are solids. At room temperatures, the liquids show broad diffuse absorption bands extending from about 2400 to 2800 A. U. In the solid state these broad diffuse bands are resolved into a number of narrower and sharper bands. Liquid toluene supercooled to liquid air temperatures shows the same absorption as the liquid at room temperatures. In the solid state, the lowering of the temperature from liquid air to liquid hydrogen temperatures produces further resolutions and greater sharpness. At liquid hydrogen temperatures the limits of the absorption bands of benzene, and ortho and para xylenes are sharp, whereas, the bands of toluene, meta xylene and mesitylene are diffuse.

Wave lengths of these bands will appear later in the Journal of the Optical Society of America.

The present paper will appear in full in J.O.S.A. & R.S.I.

15. Herbert E. Ives and A. L. Johnsrud **Bell Telephone Laboratories**

THE THICKNESS OF SPONTANEOUSLY DEPOSITED PHOTOELECTRICALLY ACTIVE RUBIDIUM FILMS, MEASURED OPTICALLY

The thickness of the thin photoelectrically active films of rubidium which spontaneously deposit on glass and platinum in high vacuum have been studied by observations on the phase shift and azimuth of polarization of reflected light. On the basis of computations made in accordance with electromagnetic theory, and using newly determined optical constants, it appears that these films are of the order of magnitude of one atom thick.

The present paper will appear in full in J.O.S.A. & R.S.I.

16. Frank Gray (Introduced by H. E. Ives) **Bell Telephone Laboratories**

THE USE OF A MOVING BEAM OF LIGHT TO SCAN A SCENE FOR TELEVISION

The paper is a discussion of a method of scanning employed in a television system. A three dimensional subject is scanned directly by a moving beam of light to produce a picture current in photoelectric cells. This method permits the use of a very intense transient illumination and more than one large aperture photoelectric cell to collect reflected light. These two factors give a highly efficient optical system for producing a picture current at transmitting station. The image seen at a distant station is the same as if light came out of the photoelectric cells to illuminate the subject and a small aperture lens formed an image of the subject for transmission. The television system transmits only the spacial variations of brightness and not the absolute brightness of the view; consequently, an additional steady illumination of a subject does not affect the reproduced image.

The present paper will appear in full in J.O.S.A. & R.S.I.

17. Arthur Edward Ruark **Mellon Institute of Industrial Research, University of Pittsburgh and Gulf Oil Companies.**
Harold C. Urey **The Johns Hopkins University**

THE IMPULSE OF THE LIGHT QUANTUM

It is shown that if angular momentum $h/2\pi$ is attributed to every light quantum regardless of its state of polarization, the selection principles for the inner quantum number can be derived. All cases of emission and absorption in a magnetic or electric field can be explained by this hypothesis coupled with the assumption that the amplitudes and phases of a macroscopic radiation field serve only to govern the statistical properties of the aggregate of quanta giving rise to the field. E.g., linearly polarized light can be composed of circularly polarized quanta if they have appropriate phases. On this basis there should be several kinds of linearly polarized light depending upon the positions of the impulse moment vectors of the individual quanta with respect to the direction of flight and the plane of polarization. Applications to the plane of polarization of resonance radiation are given and experiments to test the theory are suggested.

18. Richard Hamer and Conrad K. Rizer **University of Pittsburgh**

THE EFFECT OF SMALL CHANGES AT MODERATE TEMPERATURES ON THE ABSORPTION SPECTRUM OF IODINE

White light from a Nernst blower was passed through different absorption cells containing iodine vapor. Absorption spectra for temperatures increasing from 35 to 240°C were taken with a Hilger Littrow type quartz spectrograph. For this work two cells of different lengths previously exhausted of air and containing iodine vapor were sealed off at about 35°C. Their use enabled the effect of changes in the temperature of the iodine at low temperatures and at constant density to be studied separately from the effect of the thickness of the absorbing column.

Increase of temperature alone causes a few new bands to develop at the red end of the absorption spectrum at about 7000A. Increase of density or depth of absorbing layer causes a broadening of the continuous absorption in the blue-green end. Change of temperature showed no effect in this latter region.

These experiments indicate a simple means of studying changes in the virbrational and possibly rotational states of iodine undisturbed seriously by the possible presence of the spectrum from atomic iodine. There is thus afforded an interesting glimpse of the effect of the development of the higher quantum states of the more highly activated diatomic molecules

as their average kinetic energy is increased. A simple demonstration of the development of these bands in the red is obtained with either an open bulb or a sealed cell containing heated iodine vapor placed before a Hilger constant deviation spectroscope. By setting the cross hairs on the last visible band at about 40°C the development of 4 or 5 extra bands can be readily observed when the temperature is raised to about 180°C.

Nakamura has recently shown a development of absorption bands in the near neighborhood of 7000A as the temperature is raised from 140°C to 1000°C at constant vapor pressure.

The present work indicates that the same development takes place at a much lower pressure namely that corresponding to 35° instead of 140°C and for much smaller range of temperature change at moderate temperatures.

Bibliography:

G. Nakamura, Memoirs, College Science, Kyoto Imperial University, Series A, 9, No. 5, pp. 315–370; 1926.

The present paper will appear in full in J.O.S.A. & R.S.I.

Grundriss der Refraktions- und Brillenlehre für Studierende und Ärzte. By Prof. Dr. C. A. Hegner, Luzern. iv.+144 pages, with 94 diagrams and illustrations. Urban & Schwarzenberg, Berlin and Vienna. 1927. Mk. 7, in paper cover; Mk. 8.5, bound.

This useful little book on ophthalmological optics (as it is called in Germany) is intended primarily for the benefit of oculists, who presumably have not received a thorough training in the professional medical schools in this branch of ophthalmology and are therefore not likely to be abreast of modern progress in the study of the dioptrics of the eye and in the theory and practice of spectacle optics. Nearly all these new developments can be traced to the influence of Gullstrand and the sympathetic coöperation and practical initiative of Carl Zeiss in Jena. To the systemization and dissemination of this knowledge perhaps no one has contributed so much as Professor v. Rohr, who is the great authority in everything that pertains to the optics of the eye as also in so many other parts of optics. His writings and text-books and those of his disciples may almost be said to constitute a special department of optical literature. Thus the work before us, as the author himself is careful to state, makes no pretence to being original; it probably contains nothing that cannot be found in previous books by such German writers as v. Rohr, Henker, Brückner and Erggelet, as will be obvious at a glance to anyone who is familiar with the subject. Nevertheless, by writing this book expressly for the needs of ophthalmologists, who apparently in Germany as elsewhere have not always been affable towards the new teaching, Professor Hegner has performed a valuable service which undoubtedly needed to be done and which he was well qualified to do. He suspects that the somewhat unfriendly attitude manifested by some of his confrères in the medical profession may perhaps be due to the dread of having to discard old methods and study subjects that are not always easy to learn, simple as they may be to the initiated. Nay, some ophthal-

mologists, he says, even go so far as to belittle the new knowledge altogether, maintaining that it is of no practical value whatever so far as the training of an oculist is concerned. That this view is wholly mistaken and cannot long continue to be held, Professor Hegner says is shown by the enthusiasm and success with which many non-medical practitioners nowadays are "working themselves into the domain of ophthalmological optics;" so that in Germany also, as here in the United States, there is exactly the same issue between oculists and optometrists as to what kind of training is most essential for ocular refraction, that is, "for the measurement of the powers of vision and the adaptation of lenses for the aid thereof," as the practice of optometry is defined by law in the state of New York. It is a question of very great practical importance for the public at large, irrespective of the quarrels that have been engendered by it. In the complex environments of modern life diseased conditions of the eye are far too common; but they are rare indeed as compared with the very widespread prevalence of some form of ordinary defect of vision, such as near-sightedness, far-sightedness, astigmatism, cross eyes, etc., which can be the source of endless irritation and annoyance and may sometimes have very serious consequences, as everybody knows. It would seem to be obvious that the best experts ought to be trained for this service, and that the one thing needful, no matter what else were required, would be a thorough knowledge of physics and optics.

For the benefit of ophthalmologists and optometrists in this country and in England, it is to be hoped that it may be found possible to bring out an English translation of Professor Hegner's book.

J. P. C. S.

INSTRUMENT SECTION

SOME APPLICATIONS OF SPARK PHOTOGRAPHY

By D. B. Woodbridge and A. E. Parker

In this paper it is proposed to demonstrate the utility and simplicity of spark photography as an analytical method in the examination of various phenomena which occur too rapidly for observation by the human eye or by a camera equipped with an ultra-rapid shutter. The problem in photographing rapidly moving objects is, of course, to expose the negative for a very short space of time during which the object may be considered as stationary; and at the same time to have illumination sufficient to produce an impression on the plate. The electric spark from a condenser can be made to satisfy both these conditions to a remarkable degree. The work to be described has been done in examining the action of ping-pong balls supported by jets of air and water, the formation of drops of water and the action of soap films.

The method, in simplest terms, is one of photographing directly on the negative the shadow formed by the object placed between the photographic plate and the electric spark, which being a point source gives a clear and distinct outline. In the case of transparent objects a difference in density at a particular spot causes the light to be refracted more or less at this point and so produce a sort of shadow on the plate. This makes the method particularly useful for photographing sound waves set up by bullets.

In 1877, A. M. Worthington[1] used the light from an electric spark to observe the action of drops of water and other liquids falling on a horizontal plate. The earliest application of the electric spark to photography was that by E. Mach[2] in 1881 in taking pictures of moving projectiles. Lord Rayleigh[3] used the method for examining falling drops and jets of water and the breaking of soap films in 1891. He used a camera and lenses; an unnecessary complication which reduces the size of the image without increasing the detail. The direct shadow

[1] Worthington, Proc. Roy. Soc., *25*, p. 261 and p. 498; 1877; *34*, p. 217; 1882. Toepler, Pogg. Ann., *131*, p. 33; 1867.

[2] Mach, Sitzber. d. k. Akad. d. Wissensh. zu Wien., *98*, p. 1333; 1889.

[3] Rayleigh, Nature, *44*, pp. 249–254; 1891.

method was introduced by Boys[4] in 1893 for the photographing of bullets and is much simpler, giving, in general, much more satisfactory results. Further advances along this line have been made by Foley[5] in 1912 in his work with sound waves and Quayle[6] in 1922 with moving projectiles.

In photographing projectiles moving at a rate of as much as 2700 ft/sec and which as a result are in front of the plate for only 0.0003 second, the problem of timing the occurrence of the spark and its duration is an important one and requires more apparatus than was available for the work at this stage. In general, the undamped spark is of sufficiently short duration to give a distinct image of falling water drops (which have a speed of perhaps 4 to 6 ft. per sec when in front of the plate), and the rupture of soap films (which occurs at a rate of about 50 to 100 ft. per sec).

The apparatus used for the work with the ping-pong ball and the water drops is shown in the schematic diagram, Fig. 1. The source of

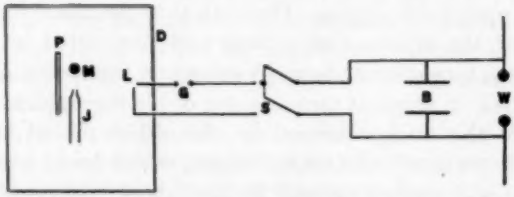

FIG. 1. *Arrangement of apparatus for photographing ping-pong balls on jets of air or water by means of electric sparks.*

electricity was a four plate Wimshurst machine *W* operated by hand. A double pole switch *S* was used to prevent leakage through the spark gun before the discharge. A stationary gap *G* in series with the spark gun *L* regulated the minimum intensity of the spark. The Leyden jar battery B had a capacity of 0.02 microfarads. The air condenser was the only one found practicable, for at the potentials used in this work a liquid dielectric such as transformer oil becomes a mechanical conductor, while the cost of mica condenser would be excessive. The spark gun is similar in design to that used by Quayle. The terminals

[4] Boys. Smithsonian Inst., pp. 165–182; 1893. Nature, *47*, pp. 415–421 and 440–446; 1893. "Soap Bubbles and the Forces Which Mould Them," London, 1890. Photographic Journal, *16*, pp. 199–209; 1891–92.

[5] Foley, Phys. Rev., *35*, p. 373; 1912.

[6] Quayle, Jour. Frank. Inst., *193*, pp. 627–640; 1922. Sci. Papers, Bu. Stands., No. 508, 1924.

consist of aluminum wire 1 mm in diameter and are adjustable for varying the size of the gap. The gun is operated simply by charging the Leyden jars and closing the switch. The actual exposure of the plates took place in a separate dark-room compartment *D*, large enough for the experimenter to work in, which contained the gun and photographic plate *P*. In both the jet of air and of water, *J* is the tube for the jet and *M* is the ping-pong ball.

In the work with soap films where it was desired to examine the effect of a ball dropping through the film, some method of timing the spark to coincide with the passage of the ball in front of the plate had to be employed. For this an adjustable spring switch *S*, Fig. 2, was

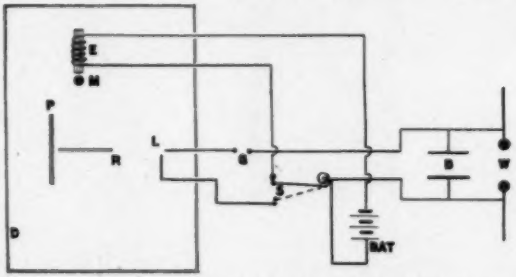

FIG. 2. *Arrangement of apparatus for photographing falling bodies which must arrive at a definite position when the spark occurs.*

used by which the interval between the breaking of the electro-magnet circuit and the closing of the spark circuit could be regulated. This interval could be determined by releasing the ball and watching the shadow on a piece of white paper substituted for the photographic plate. Difficulty sometimes arose from the fact that the time of occurrence of the spark varies with the intensity of the charge on the Leyden jars, for if the charge is strong the current will jump across the terminals of the delayed switch sooner than in the case of a weak charge. For accurate work a potential limiter such as that used by Quayle is needed. With the present apparatus much depends on the ability of the operator of the influence machine to judge the charge on the Leyden jars.

By means of the spark method it has been possible to photograph a stream of water in water, and it was thought that similar results might be obtained from a ping-pong ball on an air jet, but apparently the difference between the jet and the surrounding air was too slight to have any effect on the light from the spark. Accordingly it was found necessary to introduce some opaque substance into the stream of air

which would throw a shadow on the plate. Various smokes were tried without much success, the chief difficulty being to dispose of the smoke

FIG. 3. *Ping-pong ball on a jet of air. Pressure is less at A than at B, therefore the ball will be pulled back over the jet.*

FIG. 4. *Ping-pong ball supported on a jet of water. The ball is being rotated by the jet at the same time.*

FIG. 5. *The jet of water is continually breaking into drops, which causes the ball to fall a short distance until it is caught by the solid jet again.*

so as to avoid temporarily incapacitating the operator. The method finally used was to inject lamp black into the stream. In this manner very satisfactory results were obtained as shown in Fig. 3.

The results with the water jet, Figs. 4 and 5, were more interesting. The water is drawn around the ball away from its natural path by the action of surface tension, but at the same time the spinning of the ball produces a centrifugal action and the drops fly off, not, however, in the manner of sparks from a grindstone; for the water does not break up into drops immediately on leaving the surface of the ball, but rather goes off in streams which the forces of surface tension and gravity pull into the shape of concave parabolas. Since the drops are coming off chiefly on the side opposite the point of contact of the water jet,

FIG. 6. *A falling stream of water. The slim necks between the large sections in the continuous stream form the small drops when the stream finally breaks.*

FIG. 7. *When an electrically charged rod of sealing wax is held near a falling stream of water like that in Fig. 6 it breaks into a spray.*

there is an indication that a force is acting in a direction opposite to that of the surface tension which tends to hold the ball in the stream. It is interesting to note that the same ping-pong ball is held in the air jet where surface tension is not the main force holding it. Fig. 5 shows what happens to make the ball bob up and down, for here the jet which has been continuous in Fig. 4 has broken up into drops so that in another instant the ball would have fallen a short distance to be caught again by the continuous jet when it reformed.

Fig. 6 shows a stream of water falling from a small nozzle. It appeared to the eye as a continuous stream. This is a phenomenon which has been photographed in many ways, but shows here the applicability of spark photography to such work.

FIG. 8. *The manner in which the spray in Fig. 7 is formed by the drops falling in different trajectories.*

When an electrically charged stick of sealing wax is held near a small jet of water as in Fig. 7 it appears to break up into a spray. The reason for this appearance may be due to the fact that the small drops are pulled away from the vertical farther than the large ones and

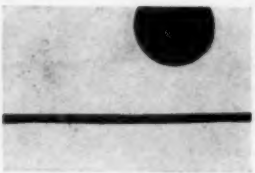

FIG. 9. *A steel ball is falling toward a ring in which a soap film is stretched.*

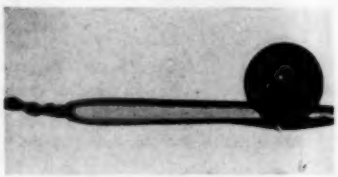

FIG. 10. *The steel ball has just made contact with the film. Note the peculiar refraction of light in the spot which should be the black shadow of the ball. This is not a chance effect.*

so there arises a series of different sized drops falling in various curves as indicated in Fig. 8. Since to the naked eye the drops in Fig. 6 appear as a continuous stream, it will be evident that Fig. 7 will appear as a

spray. An attempt has been made in Fig. 8 to show how the drops in Fig. 7 make a spray.

In the work on soap films the soap solution was made according to the formula of Boys[1]. The photographs, Fig. 9 to Fig. 15 inclusive,

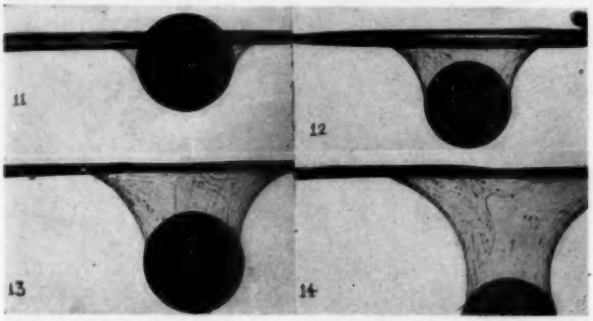

FIG. 11. *The steel ball begins to stretch the soap film*

FIG. 12. *The ball is completely through the wire loop. The film is still stretching.*

FIG. 13. *An excess of soap is carried on the bottom of the ball.*

FIG. 14. *The film is just at the point of rupture.*

are of a steel ball dropping through a soap film supported by a circular hoop about 10 cm in diameter. Two series of pictures were photographed one using a new soap solution, Figs. 9 to 15, and the other

FIG. 15. *A soap film captured in the act of breaking. The raw edges of the film may be seen distinctly in the break.*

using a solution that had been standing for a week exposed to evaporation and acquired such tenacity that a ball could be dropped through it repeatedly without breaking it, Figs. 16 to 20. In both series of films

[1] Boys, "Soap Bubbles and the Forces Which Mould Them," London, 1890

the steel ball was coated with paraffin so the film would not wet it. A wet steel ball would pass through the new solution without affect-

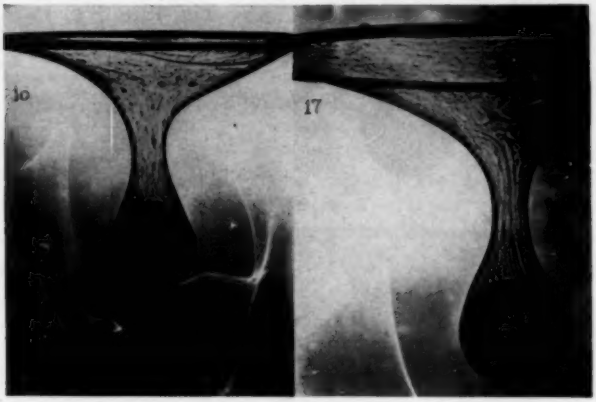

FIG. 16. *This film is tougher than the preceding ones, hence is pulled out more. Note the remarkable caustics by reflection.*

FIG. 17. *This shows further stretching of the film.*

FIG. 18. *The film has been stretched to the point where the striae have disappeared.*

ing it. The series Figs. 9 to 15 show successive stages of the ball falling through the film ending in an exposure which caught the film just in

the act of breaking. Some of the pictures show very interesting reflected and refracted caustics. The last series of pictures, Figs. 16 to 20, is interesting because it shows the steps whereby the soap film is drawn

FIG. 19. *The film healed when it contracted forming a small bubble from the slim neck of the elongation. The ball has carried a small bubble with it.*

out, pinches down and heals without breaking, even forming a small bubble between the part which goes along with the steel ball and that which remains in the hoop, Fig. 19. More elaborate apparatus is being

FIG. 20. *The film has now returned almost to its original position but still whole.*

developed with the idea of applying this method to many other physical phenomena which occur in a short space of time.

The authors wish to express their appreciation of the interest and help given during the progress of this work by Professor S. R. Williams, who suggested the problem.

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A MODIFICATION OF THE MICHELSON INTERFEROMETER

By R. M. Langer

There is a substitute for plane parallel compensating plates, which is not included among the variations of the Michelson interferometer as described in the comprehensive "Handbuch der Physikalischen Optik" by Prof. Dr. E. Gehrcke which appeared recently as part of the new edition of Winkelmann. It is convenient and simple to construct and is recommended when special arrangements have to be used. It consists of two matched prisms cemented together on their largest faces with Canada Balsam. Before cementing, one of these faces is coated with a half reflecting film. Fig. 1 shows the arrangement of totally reflecting mirrors M.

Fig. 1.

The prisms must be matched to the extent that their refractive indices must be equal for all wave lengths and the angles marked α in Fig. 2 must be closely equal. This is accomplished best by working with one stick of double height and sawing it in two after polishing.

About a dozen such cubes have been made up of glass and quartz and the following process was found to be easy after a little practice. A uniform semi-transparent film is coated on one prism and after making sure that there are no particles on the diagonal faces a drop of thick

Canada Balsam is put on one and the two prisms pressed together with as much force as one's arms can supply. The block is warmed gently on a hot plate to about 50°C turning it over and over to equalize temperatures as much as possible. In this stage the solvent of the Canada Balsam is driven off while the film is kept liquid. When the faces are all about as hot as the fingers care to handle, the face *OA* (Fig. 2) is made perpendicular to *OB*. The condition that one wishes to fulfill is that any incident ray such as *PQ* is divided into two equal rays by the face *OC* and after the separated rays have travelled over exactly equivalent paths they are joined again at some point *R* into a single ray *RS*. When this condition is satisfied, the eye will see white light fringes when

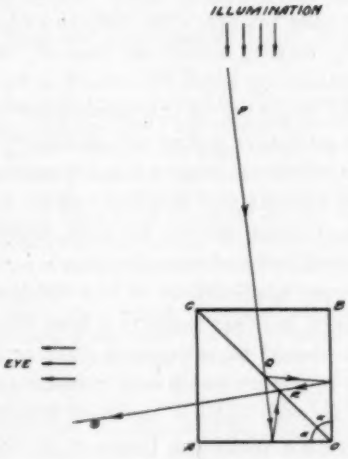

FIG. 2.

the illumination is placed as in the figure. To bring this about, a weak monochromatic source with a somewhat weaker white light background is used. Focusing the eye on the face *OB* and getting the edges at *O* superposed, the prisms are pushed around until monochromatic fringes are seen. The block is permitted to cool somewhat so that one must exert appreciable force to move the fringes. If the film becomes too stiff the block must be warmed slightly on the hot plate from time to time. Pressure is applied to make the fringes move until the white light fringes appear. The search should not take fifteen minutes. When they are in view the block is allowed to cool slowly, adjusting their position and direction at frequent intervals. When the piece has become

cold the fringes hold their position (if they have been adjusted properly while cooling) for months without much creeping.

The device has the following unusual advantages:

(a) All rays are compensated in a single piece so that repeated adjusting after each change in apparatus is unnecessary.

(b) It is the most economical in light losses since there is the minimum of air-glass transitions and all are at approximately normal incidence, and since it is unnecessary to make a heavy coat to get rid of false reflections.

(c) The semi-reflecting film is protected from deterioration by chemical or mechanical agents and the troublesome process of getting a good coat is done once for all.

(d) There are no false images. The reflections from OA and OB are easily disposed of by making α different from 45° or simply by using a telescope or other observing apparatus which is focused on the proper fringes. It is a relief to get rid of this annoyance.

(e) No holder or adjusting screws are necessary. A bit of soft wax or a light spring keeps it in place since it is a geometrically stable figure.

(f) It constitutes without any auxilliary parts a remarkably simple and rugged Michelson interferometer for some demonstration purposes. It can be held in the hand and passed around.

There may be some applications of the Michelson interferometer which it will not serve, in other respects it may be more versatile than plane parallels. It should be mentioned that crystalline prisms can produce double sets of fringes which may become mixed up under some conditions of use. In such cases it has been found useful to separate the fringe systems with a Wollaston Prism or to eliminate one set with a Nicol.

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